Couette Flows

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1 Introduction

When a fluid is between a stationary surface and a moving surface, a Couette flow is created. A 2D model could be a fluid sandwiched between stationary bottom surface and a moving top surface with velocity U, as shown in Fig.1.

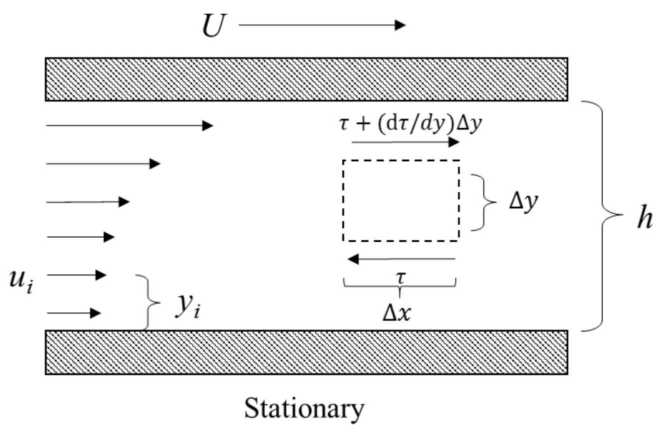


Figure 1: A 2D model of Coutte flow

The shear stress, τ, is a differentiable function of the distance from the bottom surface, denoted by y. Consider a rectangular region inside the fluid, if the shear stress at bottom of the region is τ, then the shear stress at an infinitesimal length above is $\tau + (d\tau/dy)\Delta y$. Balancing the horizontal force requires

$$0 = (\tau + \frac{d\tau}{dy}\Delta y)\Delta x - \tau\Delta x = \frac{d\tau}{dy}\Delta y\Delta x,$$

which implies

$$\frac{d\tau}{dy} = 0. \tag{1}$$

2 Newtonian fluid

Shear rate is defined as the rate at which horizontal velocity changes with respect to height. If we let $u(y)$ be the horizontal velocity as a function of height, then shear rate is du/dy. A Newtonian fluid has property that its shear stress is proportional to shear rate, which can be expressed as

$$\tau = \mu \frac{du}{dy}, \tag{2}$$

where μ is the fluid's viscosity. Combining Eq.1 and Eq.2 gives

$$\frac{d}{dy}(\mu \frac{du}{dy}) = 0,$$

which implies

$$\mu \frac{du}{dy} = c, \tag{3}$$

where c is a constant that can be determined by the boundary conditions $u(0) = 0$ and $u(h) = U$. This differential equation is easy enough to be solved using technique of separation of variables, and the result is

$$u = \frac{c}{\mu} y + k.$$

The boundary conditions $u(0) = 0$ and $u(h) = U$ require $k = 0$ and $c = U\mu/h$, so the specific solution is

$$u = \frac{U}{h} y. \tag{4}$$

The linear relationship between horizontal velocity and height is the signature of a Coutte fluid of a Newtonian fluids. However, a fluid is not necessarily Newtonian to behave Newtonian, that has linear $u(y)$. A non-Newtonian fluid may also behave Newtonian.

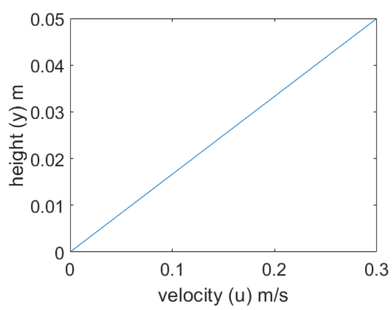
To analyze the behavior of Couette flow numerically, we can use program languages to build a numerical model, which also helps us to see how changing parameters affect the behavior of the fluid. A MATLAB function that solves Eq.3 can be written as follow

```
function [y,u] = CoutteNewtonianFluid(U,h,mu)
figure('DefaultAxesFontSize',18)
%based on Example code 9.1
[y,u]=ode45(@der,[0,h],[0],[],U*mu/h,mu); %assuming c=U*mu/h
plot(u,y)
ylabel('height (y) m')
xlabel('velocity (u) m/s')
    function uPrime=der(y,u,c,mu)
    uPrime=c/mu;
    end
end
```

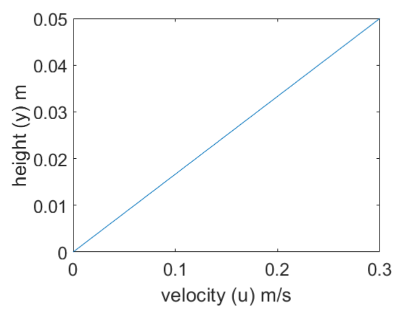
To see how different Newtonian fluids behave, let two surfaces separated by $h = 0.05\ m$, top surface moving at velocity $U = 0.3\ m/s$, and viscosity of one fluid $\mu_1 = 8.90 \times 10^{-4}\ kg/m \cdot s$, which is approximately the viscosity of water at 25 °C, and another fluid $\mu_2 = 1.50 \times 10^{-3}\ kg/m \cdot s$, which is the viscosity of mercury. Following commands

```
[yw,uw] = CoutteNewtonianFluid(0.3,0.05,8.90e-4)
[ym,um] = CoutteNewtonianFluid(0.3,0.05,1.50e-3)
```

return argurments y and u as vectors, such that $u(y_i) \approx u_i$. The solution is shown in Fig.2 and $u(y)$ is linear as expected in Eq.4. Also different Newtonian fluids behave the same in the Couette model regardless difference in their viscosity. This can be checked by command `uw-um`, which returns a zero vector.



(a) Water ($\mu = 8.90 \times 10^{-4}\ kg/m \cdot s$)



(b) Mercury ($\mu = 1.50 \times 10^{-3}\ kg/m \cdot s$)

Figure 2: The solution to Eq.3 with $h = 0.05\, m$, and $U = 0.3\, m/s$

Notice that we assume $c = U\mu/h$ is known in the code used to solve Eq.3. Yet, to get this relation we need to solve the differential equation explicitly and apply the boundary conditions. However, there are ways to solve c numerically.

One way to search for the value of c is bisection method. Treat the fluid velocity at top u_t as a function of c, i.e., $u_t(c)$. To match the boundary condition $u(h) = U$, there should be a c such that $u_t(c) = U$. If we begin the search with a proper interval $[c_1, c_2]$, and suppose in this interval $u_t(c)$ is monotone, $u_t(c_1) - U$ and $u_t(c_2) - U$ should have different sign. Then we do the same calculation at the mid point,i.e., $u_3 = (u_1 + u_2)/2$, and if $(u_t(c_1) - U)$ and $u_t(c_3) - U$ have same sign, define a new interval $[c_3, c_2]$; otherwise, define the new interval to be $[c_2, c_3]$. Repeat the process till the interval is small enough, which means either end point of the last interval is close enough to the real value. Using this method, an adapted code to solve Eq.3 can be

```
function [y,u,c] = CoutteNewtonianFluid2(U,h,mu)
%based on Example code 9.1
c1=1e-5;c2=10; %two initial guesses
tol=1e-6; %tolerance of c
err=inf;
```

```
while err>tol
    [y1,u1]=ode45(@der,[0,h],[0],[],c1,mu);
    [y2,u2]=ode45(@der,[0,h],[0],[],c2,mu);
    [y3,u3]=ode45(@der,[0,h],[0],[],(c1+c2)/2,mu);
    e1=u1(length(u1))-U;
    e2=u2(length(u2))-U;
    e3=u3(length(u3))-U;
    if sign(e3)==sign(e1)
        c1=(c1+c2)/2;
    elseif sign(e3)==sign(e2)
        c2=(c1+c2)/2;
    end
    err=abs(c1-c2);
end
plot(u3,y3)
ylabel('height (y) m')
xlabel('velocity (u) m/s')
    function uPrime=der(y,u,c,mu)
        uPrime=c/mu;
    end
y=y3;u=u3;c=c1
end
```

Another way to calculate c is Newton's method. We can estimate the derivative du_t/dc by

$$\left.\frac{du_t}{dc}\right|_c \approx \frac{u_t(c+\delta)-u_t(c)}{\delta}.$$

Adapted code with Newton's method:

```
function [y,u,c] = CoutteNewtonianFluid3(U,h,mu)
%based on Example code 9.1 and 9.2
c=150; %an initial guess
delta=1e-8; %a small change in c
tol=1e-6; %tolerance of u(h)
err=inf;
while err>tol
    [y1,u1]=ode45(@der,[0,h],[0],[],c,mu);
    [y2,u2]=ode45(@der,[0,h],[0],[],c+delta,mu);
    U1=u1(length(u1));
    U2=u2(length(u2));
    err=min(err,abs(U-U1));
    dudc=(U2-U1)/delta; %estimated derivative
    c=c-(U1-U)/dudc;
end
plot(u1,y1)
ylabel('height (y) m')
```

```
xlabel('velocity (u) m/s')
    function uPrime=der(y,u,c,mu)
        uPrime=c/mu;
    end
y=y1;u=u1;
end
```

Following commands

```
[yw1,uw1,cw1] = CoutteNewtonianFluid2(0.3,0.05,8.90e-4)
[yw2,uw2,cw2] = CoutteNewtonianFluid2(0.3,0.05,8.90e-4)
[ym1,um1,cm1] = CoutteNewtonianFluid3(0.3,0.05,8.90e-4)
[ym2,um2,cm2] = CoutteNewtonianFluid3(0.3,0.05,8.90e-4)
```

allow us to see the numerical results. The plots are same as shown in Fig.2. The calculated constants are `cw1 = 0.005340000000000, cw2 = 0.005339842005968, cm1 = 0.009000000000000, cm2 = 0.008999563535452` which are good enough approximations compared to actual c for water and mercury ($c_w = 5.34 \times 10^{-3}$ and $c_m = 9.00 \times 10^{-3}$).

3 Non-Newtonian fluid

A non-Newtonian fluid is a fluid that does not follow Newton's law of viscosity. Different non-Newtonian fluids have different relationship among shear stress, shear rate, and time. For example, a thixotropic fluid, whose viscosity reduces as increases, has a shear stress of

$$\tau = \left(\frac{1}{(u+1)e^t}\right)\frac{du}{dy}\, kg/m \cdot s^2. \tag{5}$$

From Eq.1, we have

$$\frac{d\tau}{dy} = \frac{d}{dy}\left[\left(\frac{1}{(u+1)e^t}\right)\frac{du}{dy}\right] = 0,$$

therefore

$$\frac{du}{dy} = c(u+1)e^t. \tag{6}$$

Since du/dy now depends on time, we need to calculate c for each t to meet the boundary conditions. We can still use Newton's method to find c. The code to solve Eq.6 and in addition to show the relation of $\tau/(du/dy)$ and y can be:

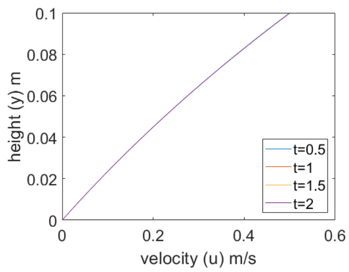
```
function CoutteNonNewtonianFluid(U,h,t)
close all
clc
%based on Example code 9.2
c=150; %an initial guess of c
delta=1e-8; %a small change in c
```

```
tol=1e-3; %tolerance of u(h)
L=[];
for i=1:length(t)
    err=inf;
    while err>tol
        [y1,u1]=ode45(@der,[0,h],[0],[],c,t(i));
        [y2,u2]=ode45(@der,[0,h],[0],[],c+delta,t(i));
        U1=u1(length(u1));
        U2=u2(length(u2));
        err=min(err,abs(U-U1));
        dudc=(U2-U1)/delta; %estimated derivative
        c=c-(U1-U)/dudc;
    end
    figure(1),plot(u1,y1)
    hold on
    L=[L,{['t=' num2str(t(i))]}];
    for k=1:length(u1)
        taududy(k)=1/((u1(k)+1)*exp(t(i)));
    end
    figure(2),plot(taududy,y1)
    hold on
end
figure(1),ylabel('height (y) m'),xlabel('velocity (u)
m/s'),legend(L,'Location','southeast')
figure(2),ylabel('height (y)
m'),xlabel('\tau/(du/dy)'),legend(L,'Location','northeast')
    function uPrime=der(y,u,c,t)
        uPrime=c*(u+1)*exp(t);
    end
figure(1),set(gca,'FontSize',18)
figure(2),,set(gca,'FontSize',18)
end
```

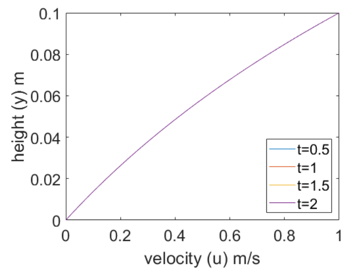
Suppose $h = 0.1\,m$, following commands gives graphs of $u(y)$ for different values of t and U:

```
CoutteNonNewtonianFluid(0.5,0.1,[0.5,1,1.5,2])
CoutteNonNewtonianFluid(1,0.1,[0.5,1,1.5,2])
CoutteNonNewtonianFluid(1.5,0.1,[0.5,1,1.5,2])
CoutteNonNewtonianFluid(2,0.1,[0.5,1,1.5,2])
```

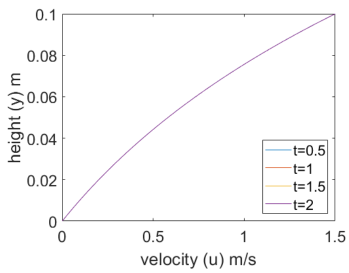
and the result is shown in Fig.3 and Fig.4. In Fig.3, curves are almost on top of each other. I suppose that is because $(du/dy) \propto e^t$, and the interval $0 < t < 2$ is just not long enough. In Fig.4, it is clear that as t increases, there is less variation in $\tau/(du/dy)$, which means the fluid behave more Newtonian.



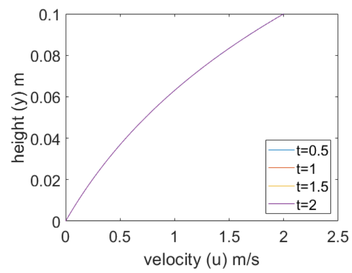
(a) U=0.5



(b) U=1.0

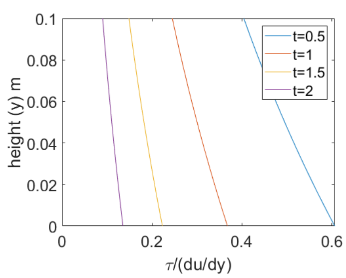


(c) U=1.5

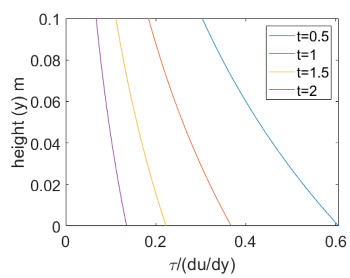


(d) U=2.0

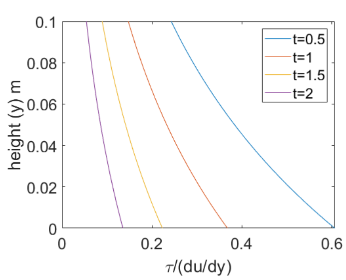
Figure 3: The solution to Eq.6 with $h = 0.1\,m$



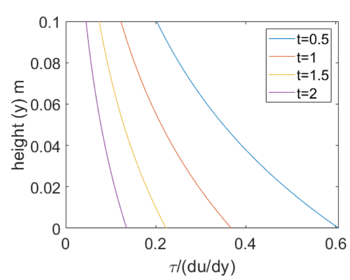
(a) U=0.5



(b) U=1.0



(c) U=1.5



(d) U=2.0

Figure 4: $\tau/(du/dy)$ versus y

4 Laminar flow

Laminar flow occurs when a fluid flows in parallel layers, with no disruption between the layers. When a fluid moving down an incline due to gravity, its shear stress satisfies

$$\tau = \gamma(h - y)sin(\theta)\, kg/m \cdot s^2, \tag{7}$$

where γ is the specific weight of the fluid and θ is the inclination angle. As with a Coutte flow, h is the thickness of the fluid, y is the perpendicular distance from the bottom surface, u is the velocity of the fluid at y. We assume $u(0) = 0$.

4.1 Laminar flow of a Newtonian fluid

From the proportionality of shear stress and shear rate of a Newtonian fluid (Eq.2) and Eq.7

$$\frac{du}{dy} = \frac{\gamma(h - y)sin(\theta)}{\mu}. \tag{8}$$

Based on this relation, we can have following code to help us analyze the laminar flow of a Newtonian fluid

```
function Laminarflow1(h,mu,gamma,theta)
close all
clc
figure('DefaultAxesFontSize',18)
hold on
L=[];
for i=1:length(theta)
    [y,u]=ode45(@der,[0,h],[0],[],mu,h,gamma,theta(i));
    plot(u,y)
    ylabel('height (y) m')
    xlabel('velocity (u) m/s')
    L=[L,{['\theta=' num2str(theta(i))]}];
end
legend(L,'Location','southeast')
    function uPrime=der(y,u,mu,h,gamma,theta)
        uPrime=gamma*(h-y)*sin(theta)/mu;
    end
end
```

SAE weight oil, which is Newtonian, and has $\gamma = 8630\, kg/m^2 \cdot s^2$ and $\mu = 0.5\, kg/m \cdot s$. If $h = 0.05\, m$, following command solves Eq.8 and the result is shown in Fig.5.

```
Laminarflow1(0.05,0.5,8630,[0,pi/24,pi/12,pi/8,pi/6])
```

Explicit solution of Eq.8 with boundary condition $u(0) = 0$ is

$$u(y) = \frac{\gamma sin(\theta)(hy - \frac{y^2}{2})}{\mu}.$$

The quadratic relationship between $u(y)$ and y gives parabola shaped curves as shown in Fig.5.

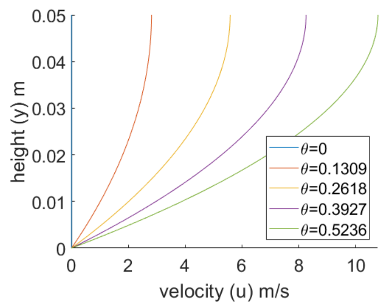


Figure 5: The solution to Eq.8 with $h = 0.05\,m$

4.2 Laminar flow of a thixotropic fluid

As mentioned before, a thixotropic fluid's viscosity reduces as time increases. From shear stress of a thixotropic fluid (Eq.5) and Eq.7

$$\frac{du}{dy} = \gamma(h-y)sin(\theta)(u+1)e^t. \tag{9}$$

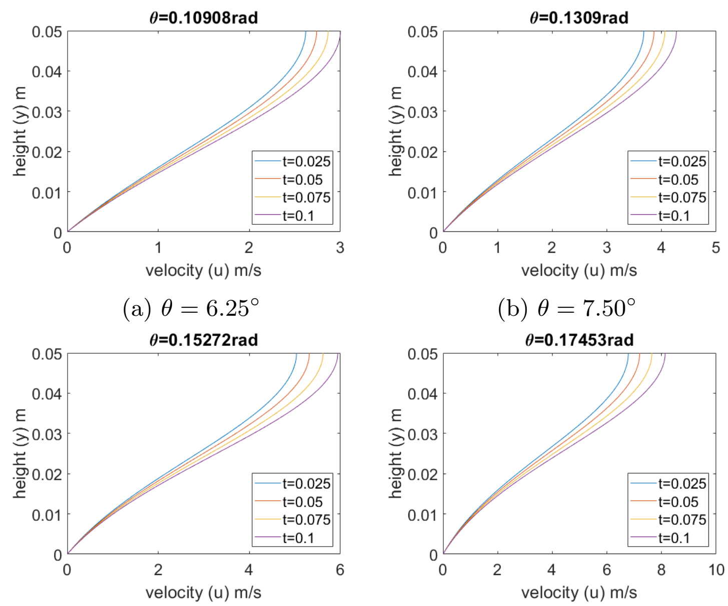
To analyze how time and angle of inclination affect behavior of a thixotropic fluid, we can use following code

```
function Laminarflow2(h,t,gamma,theta)
close all
clc
for j=1:length(theta)
    L=[];
    for i=1:length(t)
        [y,u]=ode45(@der,[0,h],[0],[],t(i),gamma,h,theta(j));
        figure(j),plot(u,y)
        hold on
        L=[L,{['t=' num2str(t(i))]}];
    end
    figure(j),title(['\theta=' num2str(theta(j)) 'rad']),
    ylabel('height (y) m'),xlabel('velocity (u) m/s'),
    legend(L,'Location','southeast'),set(gca,'FontSize',18)
end
    function uPrime=der(y,u,t,gamma,h,theta)
        uPrime=gamma*(h-y)*sin(theta)*(u+1)*exp(t);
    end
end
```

Suppose $h = 0.05\, m$ and $\gamma = 9220\, kg/m^2 \cdot s^2$, we can use command below to solve Eq.9, and graph $u(y)$ as θ varies over the interval $[5°, 10°]$ and as time varies over $(0, 0.1]$. The result is shown in Fig.6.

```
Laminarflow(0.05,[0.025,0.05,0.075,0.1],9220,
[5*(5*pi/180)/4,3*(5*pi/180)/2,7*(5*pi/180)/4,2*(5*pi/180)])
```

The relationship between $u(y)$ and y is not as simple as that of Newtonian fluid since for a thixotropic fluid, we have a nonlinear differntial equation.



(a) $\theta = 6.25°$ (b) $\theta = 7.50°$

(c) $\theta = 8.75°$ (d) $\theta = 10.00°$

Figure 6: The solution to Eq.9

4.3 Laminar flow of a rheopectic fluid

Unlike a thixotropic fluid, a rheopectic fluid has its viscosity increasing with time. Suppose a rheopectic fluid has shear stress

$$\tau = \left(15 - \frac{u}{4000}\right) + \frac{t}{40(t+1)}\frac{du}{dy}, \tag{10}$$

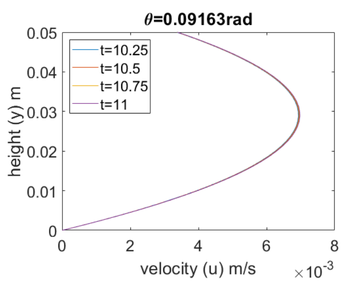
Combining Eq.7 and Eq.10 gives

$$\frac{du}{dy} = \frac{[\gamma(h-y)sin(\theta) - (15 - u/4000)]t}{40(t+1)}. \tag{11}$$

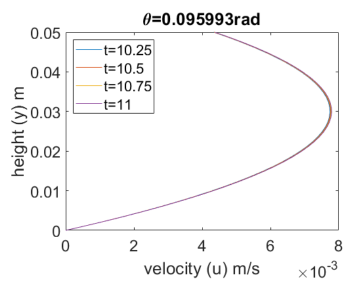
Modify the ”`function uPrime`” part of the previous MATLAB code that solves Eq.9, and we can use that to solve for Eq.11.

```
function uPrime=der(y,u,t,gamma,h,theta)
    %uPrime=gamma*(h-y)*sin(theta)*(u+1)*exp(t);
    uPrime=(gamma*(h-y)*sin(theta)-(15-u/4000))*t/(40*(t+1));
end
```

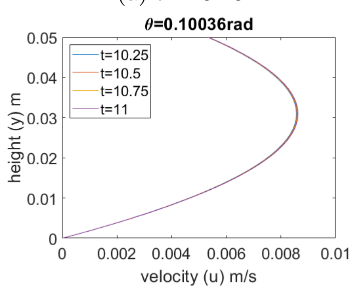
Following command plots $u(y)$ for some angles between $5° < \theta < 6°$ and as time varies between 10 and 11 seconds for a rheopectic fluid with $\gamma = 7850\,kg/m^2 \cdot s^2$ and thickness $h = 0.05\,m$, and the result is shown in Fig.7. For each θ, $u(y)$ at different time are close to each other as shown in Fig.7. This is due to the nature of the thixotropic fluid, as its viscosity increases with time. From another perspective, as t increases, $t/(t+1)$ part in Eq.10 approaches to 1, so $u(y)$ would get closer to the solution of $du/dy = [\gamma(h-y)sin(\theta) - (15 - u/4000)]/40$ as t increases.



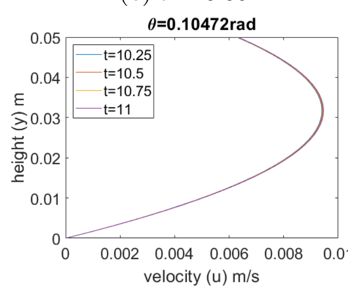
(a) $\theta = 5.25°$



(b) $\theta = 5.50°$



(c) $\theta = 5.75°$



(d) $\theta = 6.00°$

Figure 7: The solution to Eq.9