MA342 - Finite Difference Simulation of the Heat and Wave Equation

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1 Introduction

Differential equations are used extensively in engineering, mathematics, and science to model changes in system states. Two common methods used to solve partial differential equations are finite element and finite difference methods. In this report, we use finite difference methods for solving the heat and wave equations. The heat equation was used to model the heat of a 2D plate with constant temperature and constant heat flux boundary conditions. The wave equation was used to model the vibrations of a string and drum head. The heat equation simulations were written in MATLAB, and a C/C++ extension of MATLAB was used for the bulk of the computation in order to reduce the computation time for graphics generation.

2 Modeling

2.1 Heat Equation

The heat equation describes how heat flows through a medium. Let $u(x, y, t)$ be the temperature at position (x, y) at time t. Consider a differential element centered at (x, y) with area $\Delta x\Delta y$, where Δx is the width and Δy is the depth. Suppose the area is sufficiently small such that we can ignore the temperature variance within the element. Then, the total heat inside an element at time t can be expressed as

$$u(x, y, t)\Delta x\Delta y.$$

Let $\mathbf{F}_x(x, y, t)$ be the flow rate per unit time per unit length along the positive x-axis, and $\mathbf{F}_y(x, y, t)$ be the flow rate per unit time per unit length along the positive y-axis. Conservation of energy implies that the heat changes inside a differential element is approximately the total heat flow through its boundary, which gives

$$\begin{aligned}\frac{\partial\left(u\Delta x\Delta y\right)}{\partial t}(x,y,t) \approx{}& \Delta y\mathbf{F}_x\left(x-\frac{\Delta x}{2},y,t\right)-\Delta y\mathbf{F}_x\left(x+\frac{\Delta x}{2},y,t\right)\\ &+\Delta x\mathbf{F}_y\left(x,y-\frac{\Delta y}{2},t\right)-\Delta x\mathbf{F}_y\left(x,y+\frac{\Delta y}{2},t\right).\end{aligned} \tag{2.1}$$

Dividing Equation (2.1) by $\Delta x \Delta y$, and taking the limit as $\Delta x \to 0$ and $\Delta y \to 0$ gives

$$\frac{\partial u}{\partial t} + \frac{\partial \mathbf{F}_x}{\partial x} + \frac{\partial \mathbf{F}_y}{\partial y} = 0. \tag{2.2}$$

Let $\mathbf{F} = \langle \mathbf{F}_x, \mathbf{F}_y \rangle$ and $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$, which is the two-dimensional differential operator. Equation (2.2) is now equivalent to

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F} = 0. \tag{2.3}$$

The law of heat conduction, also known as Fourier's law, states that the rate of heat transfer through a material is proportional to the negative gradient of temperature. Fourier's law in differential form is

$$\mathbf{F} = -\alpha \nabla u, \tag{2.4}$$

which gives

$$\frac{\partial u}{\partial t} - \alpha \left(\nabla \cdot \nabla u \right) = 0$$

by substituting Equation (2.4) into Equation (2.3). The coefficient α is a material property represented as

$$\alpha = \frac{\kappa}{c\rho},$$

where ρ, c, and κ represent the density, heat capacity, and thermal conductivity, respectively. We need to specify an initial condition and boundary conditions to guarantee a unique solution, where a general initial condition is described as

$$u(x, y, 0) = f(x, y).$$

There are three main types of boundary conditions Dirichlet, Neumann, and Robin. To describe a fixed temperature at the boundary, we have a Dirichlet boundary condition, which is generally represented as

$$u(x, y, t) = g(x, y, t)$$

such that for all $x, y \in \partial\Omega$ and $t \geq 0$. To describe the flux through the boundary we have a Neumann boundary condition

$$\mathbf{F}(x, y, t) \cdot n(x, y) = -\alpha \left(\nabla u(x, y, t) \cdot n(x, y) \right)$$

such that for all $x, y \in \partial\Omega$ and $t \geq 0$, where $n(x, y)$ is the unit normal vector at (x, y) for region Ω.

2.1.1 Solving Heat Equation using Finite Differences

We use a forward difference scheme to approximate the first order derivative

$$\frac{\partial u}{\partial t}(x,y,t) \approx \frac{u(x,y,t+\Delta t) - u(x,y,t)}{\Delta t}. \tag{2.5}$$

Central difference schemes are used to approximate the second order derivatives

$$\frac{\partial^2 u}{\partial^2 x}(x,y,t) \approx \frac{u(x+\Delta x,y,t) - 2u(x,y,t) + u(x-\Delta x,y,t)}{\Delta x^2} \tag{2.6}$$

and

$$\frac{\partial^2 u}{\partial^2 y}(x,y,t) \approx \frac{u(x,y+\Delta y,t) - 2u(x,y,t) + u(x,y-\Delta y,t)}{\Delta y^2}. \tag{2.7}$$

We define evenly spaced nodes $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ over a region with length L and width W, which satisfies

$$\begin{aligned} x_1 = 0 \quad &\text{with} \quad x_{i+1} = x_i + \Delta x \\ y_1 = 0 \quad &\text{with} \quad y_{j+1} = y_j + \Delta y \end{aligned}$$

and

$$t_0 = 0 \quad \text{with} \quad t_{k+1} = t_k + \Delta t$$

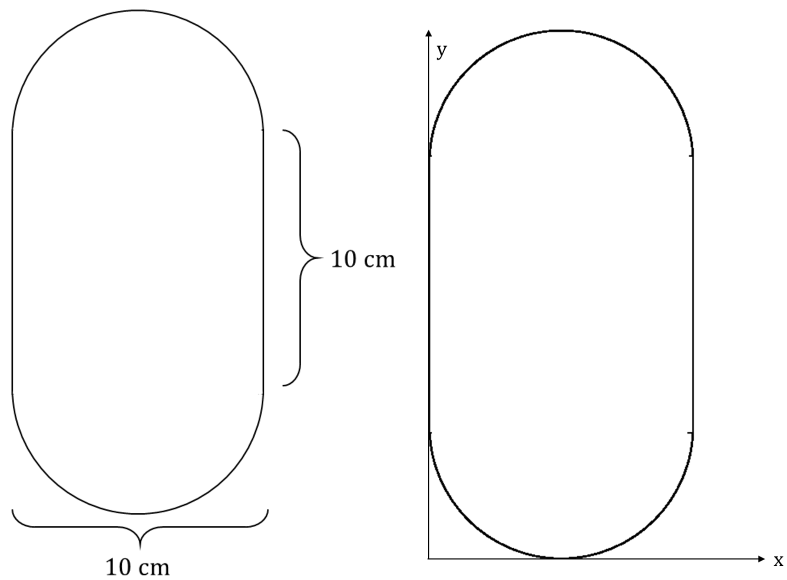
for all $i \in \{1,2,\ldots,m\}$, $j \in \{1,2,\ldots,n\}$, and $k \geq 0$. Defining $u(x_i, y_j, t_k) = u_{i,j,k}$ and combining Equations (2.5)-(2.7) gives

$$\begin{aligned} u_{i,j,k+1} = u_{i,j,k} &+ \left(\frac{k\Delta t}{c\rho\Delta x^2}\right)(u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}) \\ &+ \left(\frac{k\Delta t}{c\rho\Delta y^2}\right)(u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}). \end{aligned} \tag{2.8}$$

We can approximate a new temperature of the element at a forward time step of Δt if the temperature of an element and its four neighbouring elements are known. However, the neighboring elements at the boundary have a missing neighbor, which then shows the necessity for a boundary condition. For a Dirichlet boundary condition, we can simply fix the boundary elements. For a Neumann boundary condition, we use a difference scheme to supply the missing neighbor at the boundary. Stability needed for the model is given by

$$\Delta t < \frac{c\rho\Delta x^2 \Delta y^2}{2k\left(\Delta x^2 + \Delta y^2\right)}.$$

Using a two dimensional region as shown in Figure 1a, suppose we have the initial condition that has fixed temperature of 60°C, the Dirichlet boundary condition $u(x,y,t) = 80°\text{C}$ on the left and right boundaries, and the Neumann boundary condition $\frac{\partial u}{\partial n} = s - 1$ on the hemispherical top and bottom as shown in Figure 1.



(a) (b)

Figure 1: A two dimensional region.

The finite difference scheme describes the boundary conditions as

$$u_{1,j,k} = u_{m,j,k} = 80$$

for $10 \le y_j \le 15$,

$$u_{i-1,j,k} = u_{i+1,j,k} + \Delta x \left(\frac{x-5}{\sqrt{(x-5)^2 + (y-5)^2}} \right)$$

$$u_{i,j-1,k} = u_{i,j+1,k} + \Delta x \left(\frac{x-5}{\sqrt{(x-5)^2 + (y-5)^2}} \right)$$

for $y = 5 - \sqrt{10x - x^2}$ and $x \le 5$,

$$u_{i+1,j,k} = u_{i-1,j,k} - \Delta x \left(\frac{x-5}{\sqrt{(x-5)^2 + (y-5)^2}} \right)$$

$$u_{i,j-1,k} = u_{i,j+1,k} + \Delta x \left(\frac{x-5}{\sqrt{(x-5)^2 + (y-5)^2}} \right)$$

for $y = 5 - \sqrt{10x - x^2}$ and $x > 5$,

$$u_{i-1,j,k} = u_{i+1,j,k} + \Delta x \left(\frac{x-5}{\sqrt{(x-5)^2 + (y-10)^2}} \right)$$

$$u_{i,j+1,k} = u_{i,j-1,k} - \Delta x \left(\frac{x-5}{\sqrt{(x-5)^2 + (y-10)^2}} \right)$$

for $y = 10 - \sqrt{10x - x^2}$ and $x \leq 5$,

$$u_{i+1,j,k} = u_{i-1,j,k} - \Delta x \left(\frac{x-5}{\sqrt{(x-5)^2 + (y-10)^2}} \right)$$
$$u_{i,j+1,k} = u_{i,j-1,k} - \Delta x \left(\frac{x-5}{\sqrt{(x-5)^2 + (y-10)^2}} \right)$$

for $y = 10 - \sqrt{10x - x^2}$ and $x > 5$. Notice that some boundary conditions such as $y = \left(5 - \sqrt{10x - x^2}\right)$ may never be satisfied. When programming we used the MATLAB command `abs(Y - (5 + sqrt(-X.^2 + 10 * X))) < dy / 1.5` instead. Using the given boundary condition above and Equation (2.8), we can estimate the temperature change inside the region over time.

2.2 Wave Equation

The wave equation is the other well known partial differential equation, which is

$$\frac{\partial^2 u}{\partial t^2} = \alpha \left(\nabla \cdot \nabla u \right). \tag{2.9}$$

The α in Equation (2.9) is the speed of the wave. The wave equation is widely used for varying applications such as modeling a vibrating string and electromagnetic waves. Our first wave equation used is

$$\frac{\partial^2 u}{\partial t^2} = \frac{H}{\rho} \frac{\partial^2 u}{\partial x^2} - \kappa \frac{\partial u}{\partial t}, \tag{2.10}$$

which is almost equivalent to Equation (2.9). The difference is that Equation (2.10) has a damping term scaled by κ. The H is the common value to force equality in the derivation and ρ is the linear density of the wire at position x. These two values, H and ρ, are equivalent to the scalar α in Equation (2.9). The initial conditions for this model are

$$u(x,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = e^{-10\left(x - \frac{L}{2}\right)^2} - e^{-10\left(\frac{L}{2}\right)^2}$$

with Dirichlet boundary conditions

$$u(0,t) = u(L,t) = 0$$

for all t. For this model we also have that the length of the region, $L = 2$.

The next equation we used is the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = 2 \left(\nabla \cdot \nabla u \right), \tag{2.11}$$

where $\alpha = 2$. The initial conditions for this model are

$$u(x,y,0) = \frac{||(x,y)||_p}{10} \quad \text{and} \quad \frac{\partial u}{\partial t}(x,y,0) = 0$$

with Dirichlet boundary conditions

$$u(x, y, t) = 0.1$$

for all x and y such that

$$||(x, y)||_p = 1.$$

This was done for p-norms with $p = 1$, $p = 2$, $p = 3$, and $p = \infty$.

2.2.1 Finite Difference Methods

Similarly to the heat equation we use difference equations to generate a numeric solution to Equation (2.10). The damping term is approximated by a forward difference

$$\frac{\kappa \partial u}{\partial t} \approx \kappa \left(\frac{u_{i,j+1} - u_{i,j}}{\Delta t} \right),$$

where κ is a damping parameter. The second order derivatives are approximated by second order centered difference equations

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta t^2}$$

and

$$\frac{H}{\rho} \frac{\partial^2 u}{\partial x^2} \approx \frac{H}{\rho} \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \right).$$

The approximation of Equation (2.10) from the difference equations is

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta t^2} = \frac{H}{\rho} \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \right) - \kappa \left(\frac{u_{i,j+1} - u_{i,j}}{\Delta t} \right),$$

which gives

$$u_{i,j+1} = \frac{\left(2 - \frac{2H\Delta t^2}{\rho \Delta x^2} + k\Delta t\right) u_{i,j} + \frac{H\Delta t^2}{\rho \Delta x^2} \left(u_{i+1,j} + u_{i-1,j}\right) - u_{i,j-1}}{1 + \kappa \Delta t}.$$

The first initial condition is approximated as

$$u_{i,0} = 0$$

for all $i \in \{1, 2, \ldots, m\}$, where 1 and n are the indices at $x = 0$ and $x = L$, respectively. The second initial condition is approximated by a forward difference equation

$$\frac{u_{i,1} - u_{i,0}}{\Delta t} = e^{-10\left(x_i - \frac{L}{2}\right)^2} - e^{-10\left(\frac{L}{2}\right)^2},$$

which gives

$$u_{i,1} = u_{i,0} + \Delta t \left(e^{-10\left(x_i - \frac{L}{2}\right)^2} - e^{-10\left(\frac{L}{2}\right)^2} \right)$$

for all $i \in \{1, 2, \ldots, m\}$. The Dirichlet boundary conditions become

$$u_{1,j} = u_{n,j} = 0$$

for all $j \geq 0$. The parameter κ is determined by a bisection algorithm, which starts at values $a = 0$ and $b = 10$. This algorithm performs bisection for elapsed times of $[1, 1000]$ in steps of 1.

The general wave equation in Equation (2.9) is done in a similar procedure, where second derivatives are approximated by a second order centered difference equation. The wave equation in two-dimensions is

$$\frac{\partial^2 u}{\partial t^2} = \alpha \left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t^2} \right),$$

which becomes

$$\frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k}}{\Delta t^2} = \alpha \left(\frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{\Delta x^2} + \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{\Delta y^2} \right)$$

in terms of difference equations. From here we simplify to get the next iteration in terms of the current and previous values

$$\begin{aligned} u_{i,j,k+1} &= 2\left(1 - \frac{\alpha \Delta t^2}{\Delta x^2} - \frac{\alpha \Delta t^2}{\Delta y^2}\right) u_{i,j,k} \\ &\quad + \alpha \Delta t^2 \left(\frac{u_{i+1,j,k} + u_{i-1,j,k}}{\Delta x^2} + \frac{u_{i,j+1,k} + u_{i,j-1,k}}{\Delta y^2} \right) - u_{i,j,k-1}, \end{aligned}$$

but in our case $\alpha = 2$ so we have

$$\begin{aligned} u_{i,j,k+1} &= 2\left(1 - \frac{2\Delta t^2}{\Delta x^2} - \frac{2\Delta t^2}{\Delta y^2}\right) u_{i,j,k} \\ &\quad + 2\Delta t^2 \left(\frac{u_{i+1,j,k} + u_{i-1,j,k}}{\Delta x^2} + \frac{u_{i,j+1,k} + u_{i,j-1,k}}{\Delta y^2} \right) - u_{i,j,k-1}. \end{aligned}$$

The first initial condition is approximated as

$$u_{i,j,0} = \frac{\left(x_i^p + y_j^p\right)^{\frac{1}{p}}}{10}$$

for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. The second initial condition is approximated by a forward difference equation

$$\frac{u_{i,j,1} - u_{i,j,0}}{\Delta t} = 0,$$

which gives

$$u_{i,j,1} = u_{i,j,0}$$

for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. The Dirichlet boundary conditions are a little more complicated now, but we have

$$u_{i,j,k} = 0.1$$

for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$ such that

$$\left(x_i^p + y_j^p\right)^{\frac{1}{p}} = 1.$$

Equivalently, we have

$$x_i^p + y_j^p = 1$$

because $\lim\limits_{p \to \infty} 1^p = 1$.

3 Results

3.1 Heat Equation

First, the main geometry and boundary condition geometries were generated for the 2D plate. We discretize the bounding box of the "arena" shaped plate and used the equations of the upper and lower half circles to identify all of the discretized points contained by the main geometry. All points within the geometry were set to a value of 1 as shown in Figure 2.

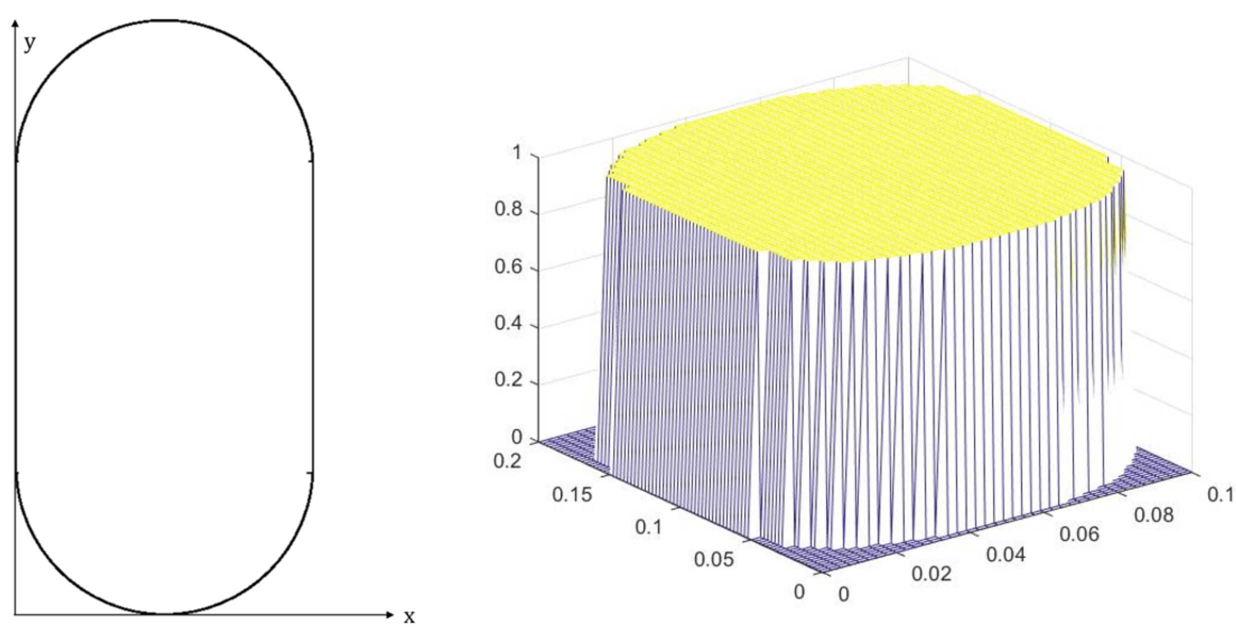


Figure 2: Discretization of the heated plate corresponding to the defined continuous geometry on the left. All length units are in meters.

The points on the edges of the geometry were then separated into the two boundary condition types. The points on the left and right of the geometry were assigned the Dirichlet condition. The curved top and bottom edges are more complicated to define. We used two different techniques for defining the points that make up the curved boundaries. The first technique used was taken from Chapter 11 [2] of the class notes, which identified edge points using the inequality

$$|x_i^2 + y_i^2 - 1| < \frac{\min\{\Delta x, \Delta y\}}{1.5}.$$

This technique will be referred to as Method 1 or the proximity method.

The other technique involved looping through every row and column, and adding the minimum and maximum index point that exists in the geometry to the boundary condition. This method will make the thinnest boundary possible while guaranteeing there are no holes along the edge. This technique will be referred to as Method 2 or the max/min method. Figure 3 shows the boundary profiles created by both methods.

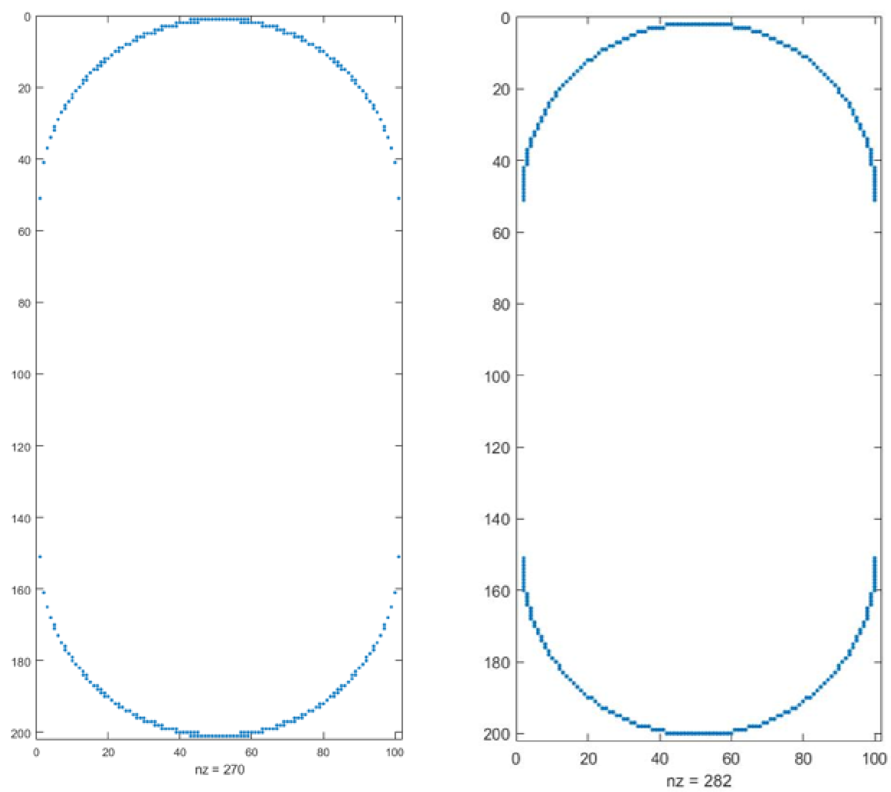


Figure 3: Boundary determination Methods 1 (left) and 2 (right) are shown visually. Method 2 included 12 additional points that were not included by Method 1.

Method 2 uses more points for the boundaries and ensures there are no holes along the boundaries. So, Method 2 was used for the rest of the results. If the boundary is determined to be too thin when compared to experimental results, Method 1 can easily be modified to increase the thickness of the boundary.

A simulation was run for 10 seconds for a variety of metals. The properties of the metals used are shown in Table 1. The copper simulation is shown in Figure 4 at times of 0.5 seconds, 2 seconds, 5 seconds, and 10 seconds. A resolution of 0.25 cm was used for both the x and y directions. The simulation of the other metals are in the Appendix.

Material	Density(ρ) $[kg/m^3]$	Conductivity (k) $[W/m^{-1}K^{-1}]$	Specific Heat (c) $[J/kg^{-1}K^{-1}]$
Copper, Pure	8954	386	380
Aluminum 6061 Temper-O	2710	180	1256
Carbon Steel 1.5%	7753	36	486
Gold, Pure	18900	318	130
Silver, Pure	10510	418	230

Table 1: Material properties of metals simulated.[1]

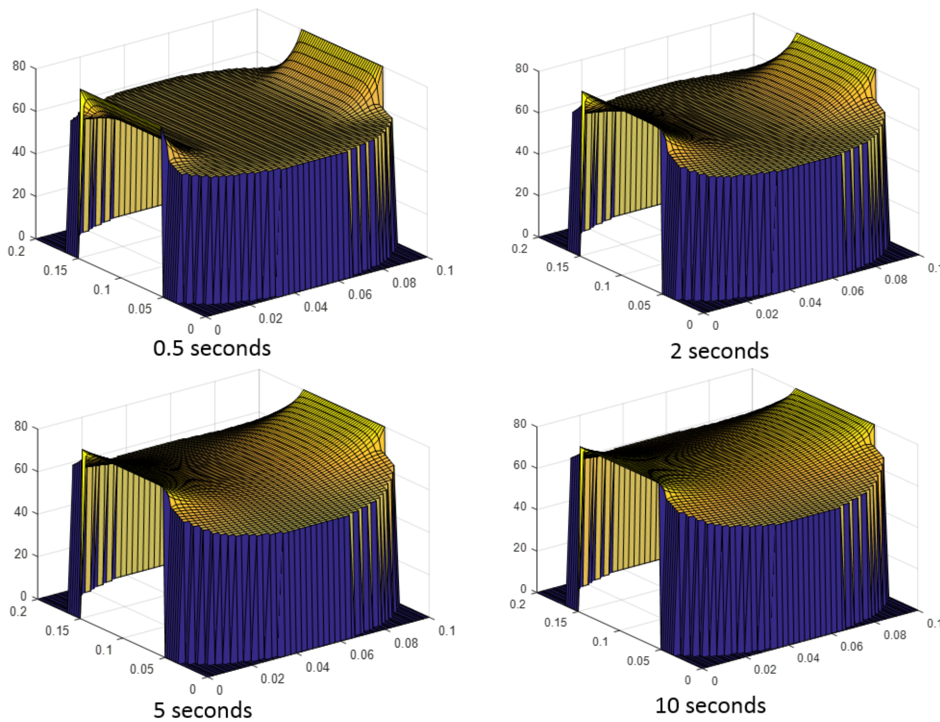
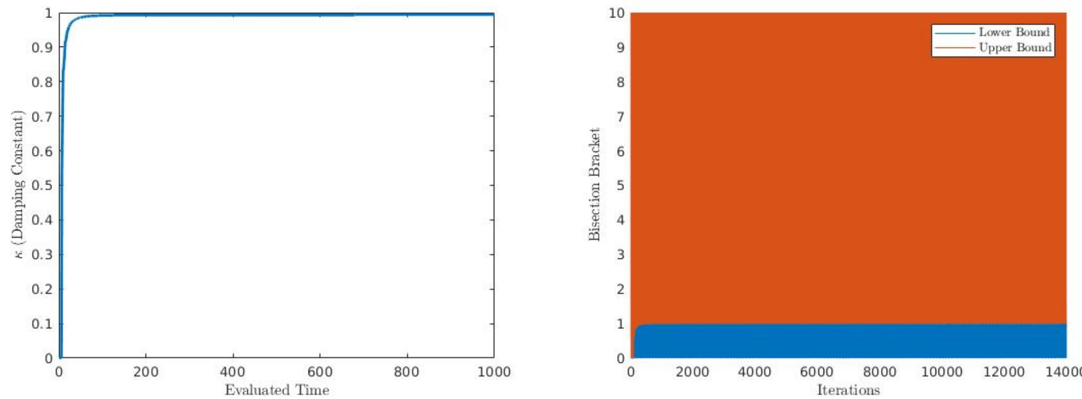


Figure 4: Simulation of the temperature gradient of a piece of copper over 10 seconds.

3.2 Wave Equation

The first result is for the smallest value of κ for which the one-dimensional wave equation remains nonnegative. This result assumes the parameters are $H = 0.1, \rho = 1$, and $L = 2$. Figure 5 shows the convergence of the κ value for Equation (2.10).



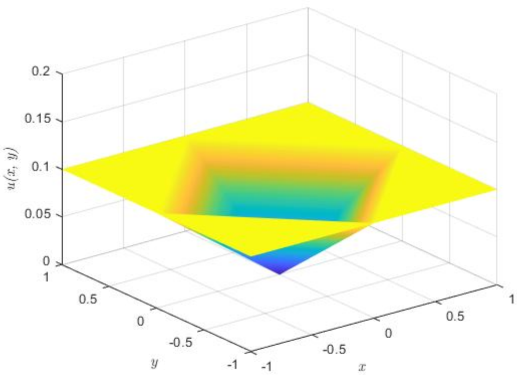
(a) κ convergence as a function of time.

(b) Bisection upper and lower bounds.

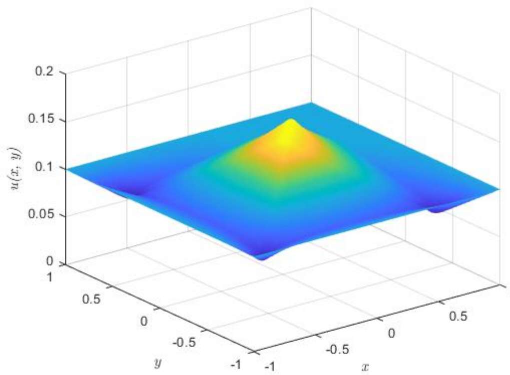
Figure 5: The κ convergence for nonnegative damping using Bisection.

The κ parameter converges to 1 as $T \to \infty$ as shown in Figure 5a. In Figure 5b, the bisection bracket is shown for each iteration of the finite difference approximation throughout all values of T. This convergence is because the solution could become negative at a larger duration when a shorter duration could have just not gone long enough.

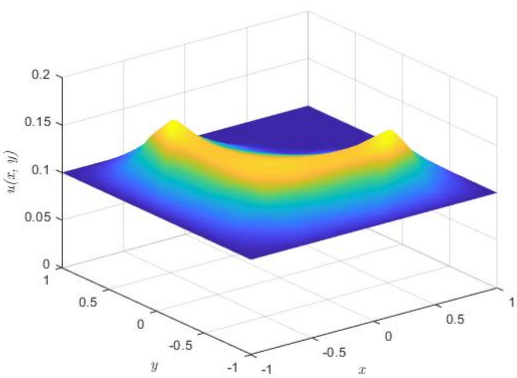
The next result is for the two-dimensional wave equation in Eqution (2.9), where $\alpha = 2$ as described in Section 2.2. Figures 6-9 show the solutions from the finite difference methods with varying p-norm values.



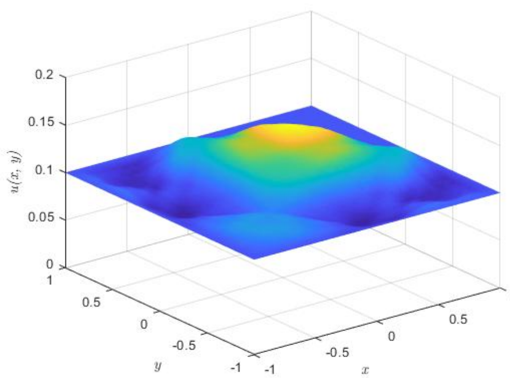
(a) Solution at $t = 0$



(b) Solution at $t = 0.5$

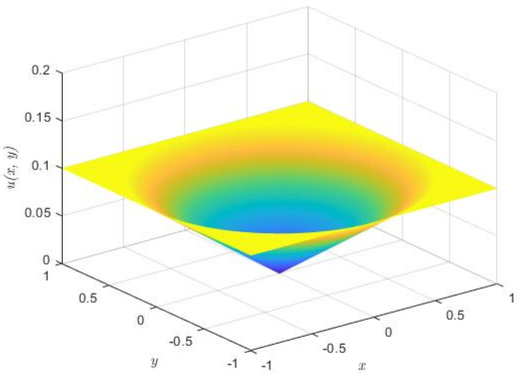


(c) Solution at $t = 1$

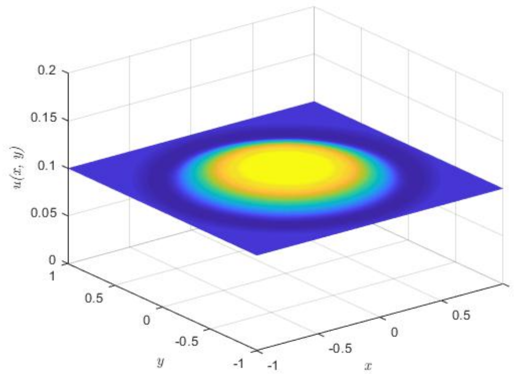


(d) Solution at $t = 5$

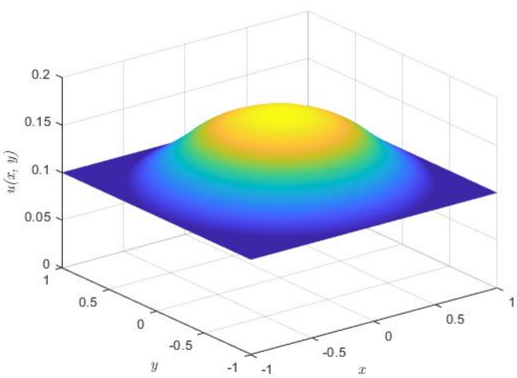
Figure 6: Wave equation over the 1-norm at various times.



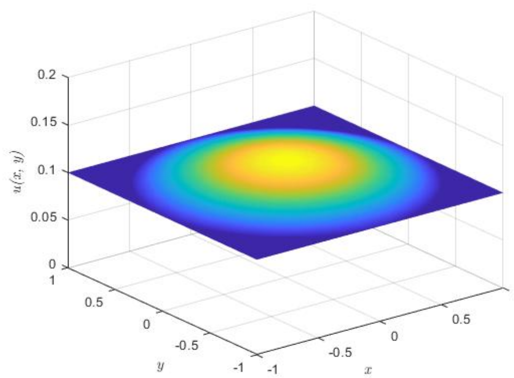
(a) Solution at $t = 0$



(b) Solution at $t = 0.5$

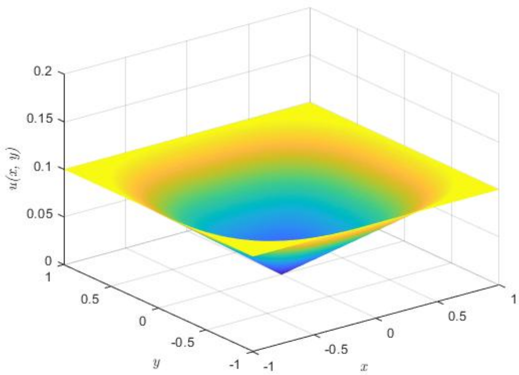


(c) Solution at $t = 1$

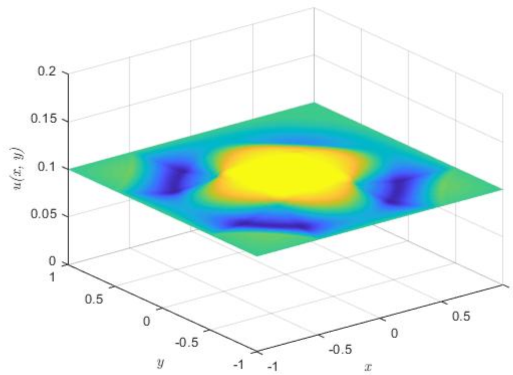


(d) Solution at $t = 5$

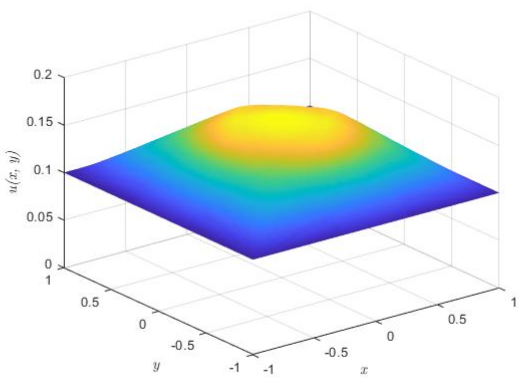
Figure 7: Wave equation over the 2-norm at various times.



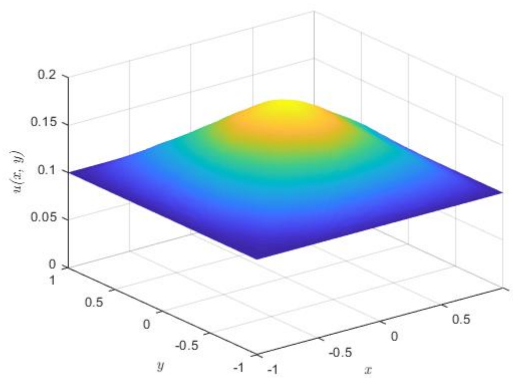
(a) Solution at $t = 0$



(b) Solution at $t = 0.5$

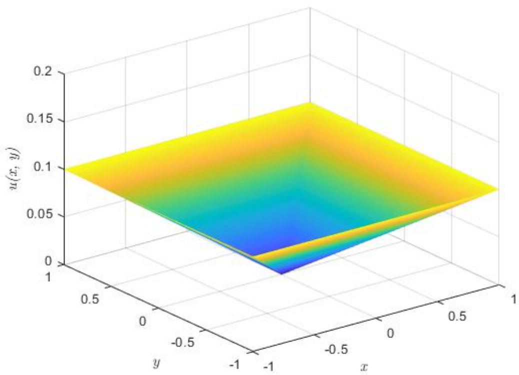


(c) Solution at $t = 1$

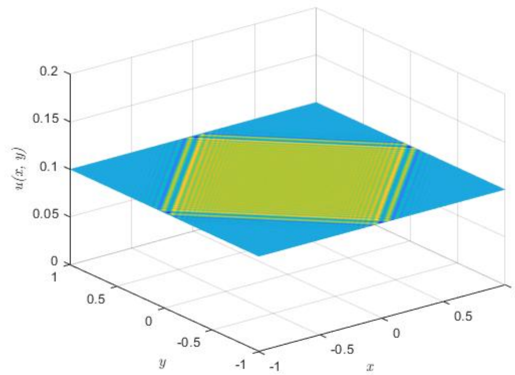


(d) Solution at $t = 5$

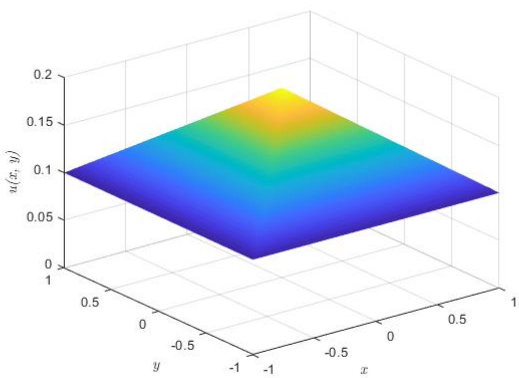
Figure 8: Wave equation over the 3-norm at various times.



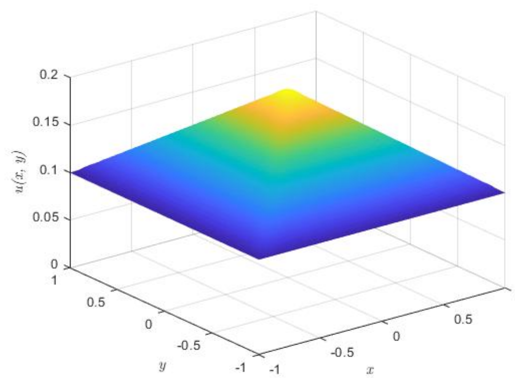
(a) Solution at $t = 0$



(b) Solution at $t = 0.5$



(c) Solution at $t = 1$



(d) Solution at $t = 5$

Figure 9: Wave equation over the ∞-norm at various times.

The solution at time, $t = 0$, shows the progression of the unit p-norm over the region.

References

[1] "Thermal Properties of Metals, Conductivity, Thermal Expansion, Specific Heat." Engineers Edge. 2019

[2] Holder, Allen. "Chapter 11. Partial Differential Equations." pp. 403-416

Appendix

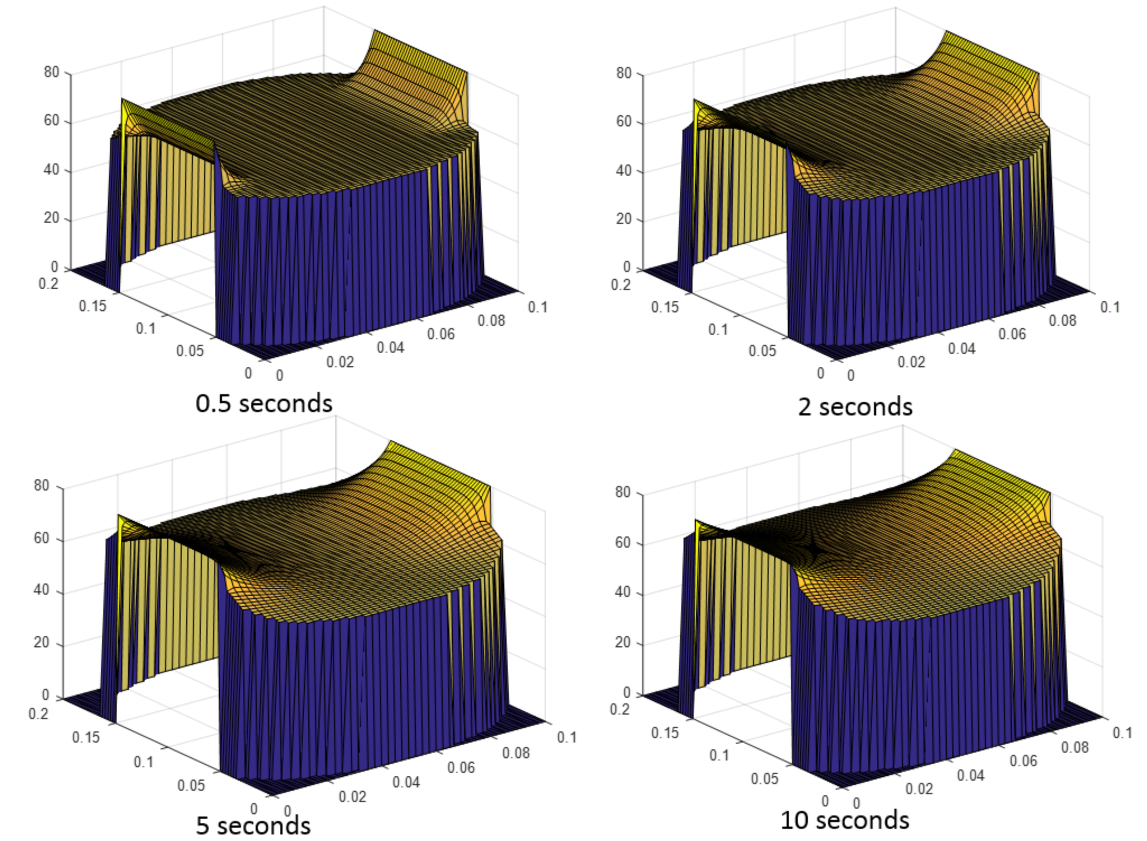


Figure 10: Simulation of the temperature gradient of a piece of aluminum over 10 seconds.

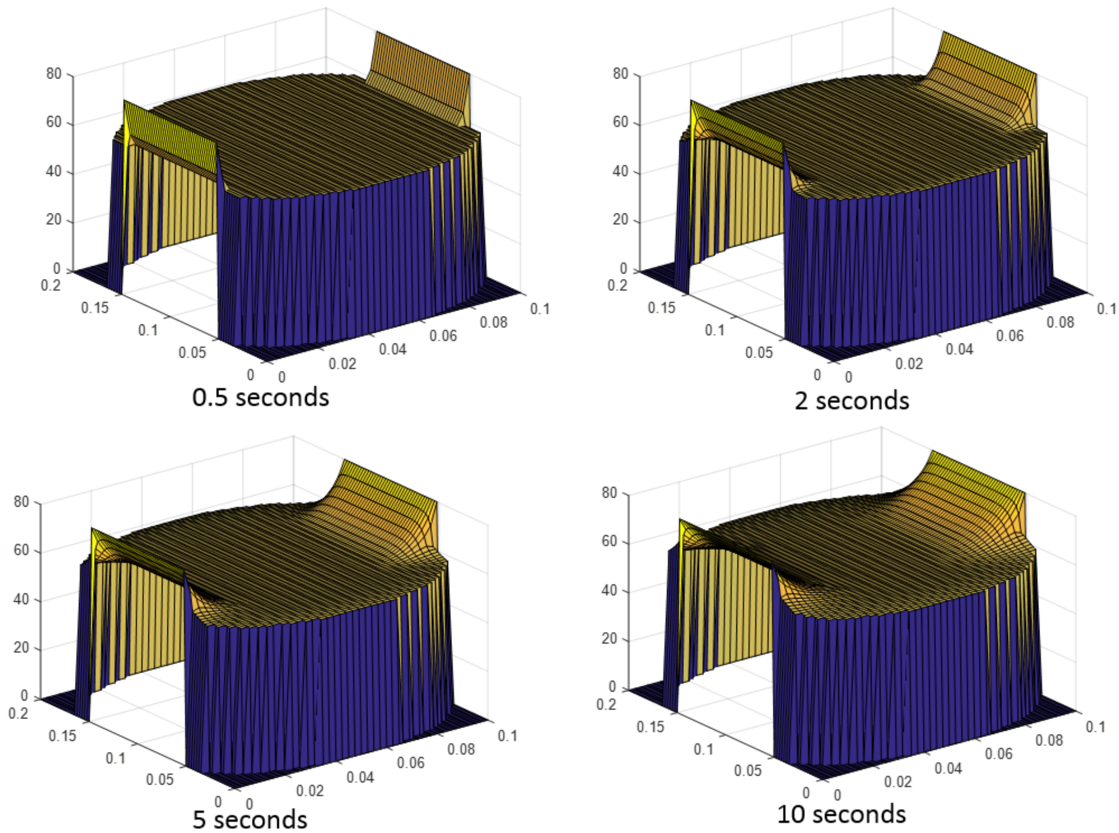


Figure 11: Simulation of the temperature gradient of a piece of steel over 10 seconds.

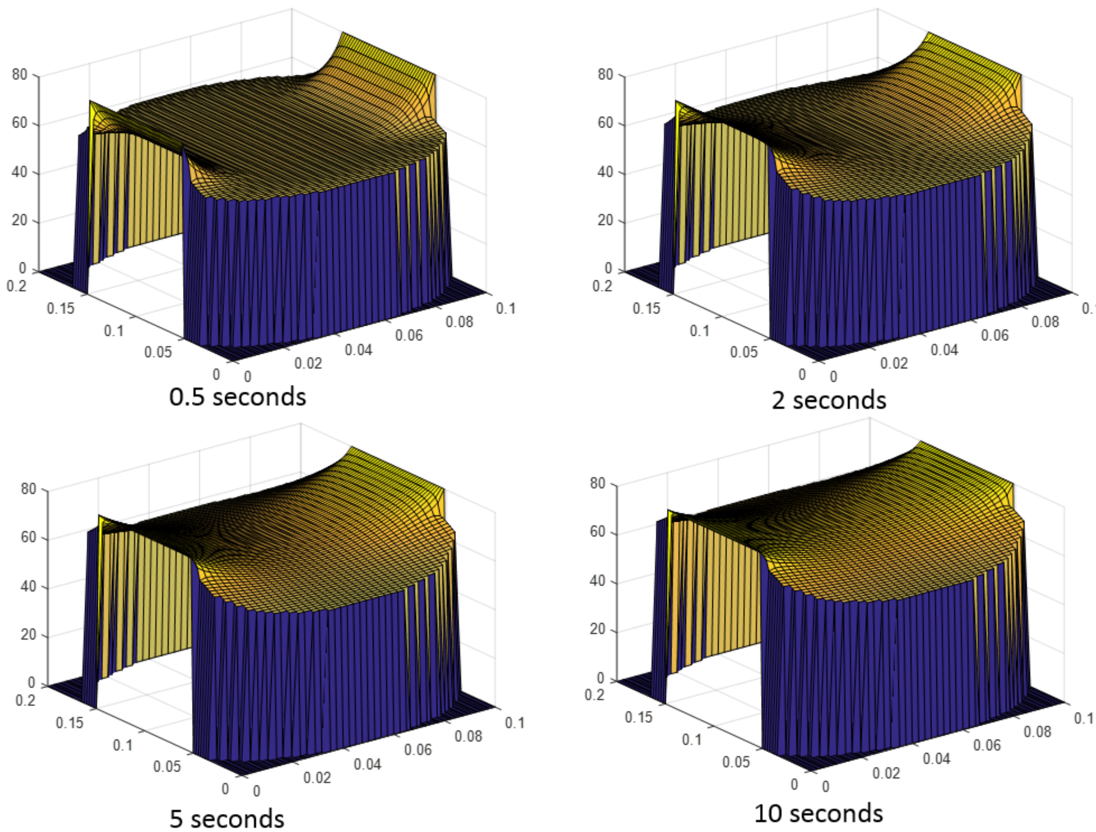


Figure 12: Simulation of the temperature gradient of a piece of gold over 10 seconds.

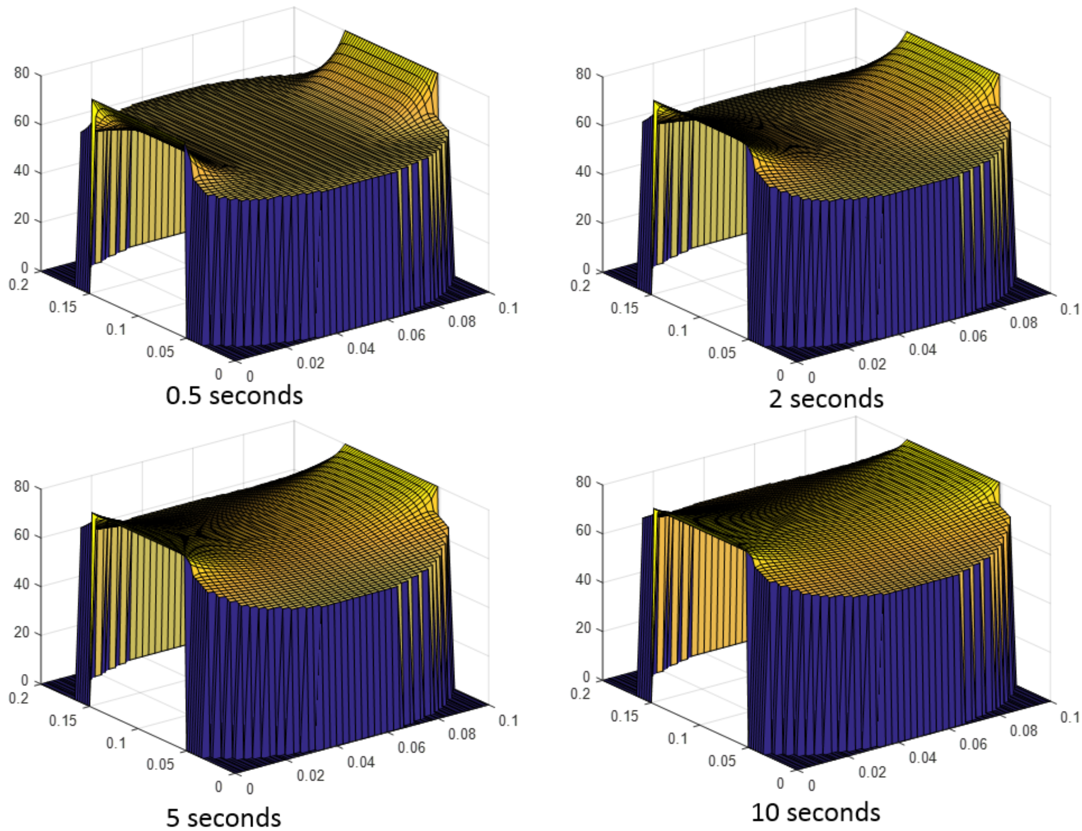


Figure 13: Simulation of the temperature gradient of a piece of silver over 10 seconds.