Numerical Method for Electric Potential Field

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The Usual Method to Solve Electric Potential Field

In electrostatics, according to Coulomb's law, the electric field, E, of a stationary charge follows the equation:

$$E(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(r')\,d\tau'$$

(ε_0 is permittivity of free space, $\hat{\mathscr{r}}$ is unit vector from charge to the point where E is measured, $\mathscr{r}$ is distance from charge to the point where E is measured, and ρ is charge density)

The potential, V, is then given by:

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}}\rho(r')\,d\tau'$$

It is easy to prove that, the potential, V, can be written as differential form, using Poisson's equation,

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho$$

When the charge inside a system is zero, Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0$$

If V depends on two variables, Laplace's equation becomes:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

To separate variables, let:

$$\mathrm{V(x,y) = X(x)Y(y)}$$

Putting the above equation into Laplace's equation, we obtain:

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$$

Let $k^2 = \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$, we then have:

$$\frac{d^2X}{dx^2} = k^2X, \frac{d^2Y}{dy^2} = -k^2Y$$

The general solution for previous differential equation is:

$$X(x) = Ae^{kx} + Be^{-kx}, Y(y) = Csin(ky) + Dcos(ky)$$

so,

$$\boxed{V(x,y) = (Ae^{kx} + Be^{-kx})(Csin(ky) + Dcos(ky))}$$

Now we can use the above equation with boundary condition to find potential of different charge distribution.

Numerical Method

If we have Poisson's equation in boundary $\{x, y \Rightarrow | \, a < x < b, c < y < d\}$:

$$f(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Let step size:

$$h = \frac{b-a}{n}, k = \frac{c-d}{m}$$

Using Taylor series, we can get:

$$\frac{\partial^2}{\partial x^2} u(x_i, y_j) := \frac{u(x_{i+1}, y_j) - 2\,u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} - \frac{h^2}{12} \cdot \frac{\partial^4}{\partial x^4} u(\alpha_1, y_j);$$

$$\frac{\partial^2}{\partial y^2} u(x_i, y_j) := \frac{u(x_i, y_{j+1}) - 2\,u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} - \frac{k^2}{12} \cdot \frac{\partial^4}{\partial y^4} u(x_i, \beta_1);$$

Substitute these two equations back to Poisson's equation:

$$2\left(1 + \frac{h^2}{k^2}\right) u(x_i, y_j) - \left(u(x_{i+1}, y_j) + u(x_{i-1}, y_j)\right) - \frac{h^2}{k^2}\left(u(x_i, y_{j+1}) + u(x_i, y_{j-1})\right) := -h^2 f(x_i, y_j);$$

When we have step size in x direction and y direction the same ($h = k$):

$$4\,u(x_i, y_j) - u(x_{i+1}, y_j) - u(x_{i-1}, y_j) - u(x_i, y_{j+1}) - u(x_i, y_{j-1}) = -h^2 f(x_i, y_j);$$

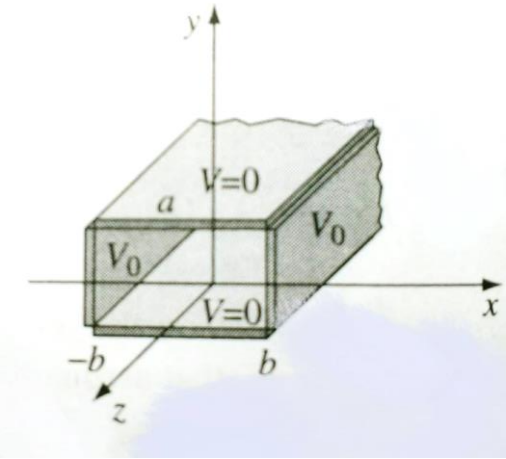
If we have Laplace's equation instead of Poisson's equation ($f(x,y) = 0$):

$$4\,u(x_i, y_j) - u(x_{i+1}, y_j) - u(x_{i-1}, y_j) - u(x_i, y_{j+1}) - u(x_i, y_{j-1}) = 0$$

In a grid, the above equation would indicate that value of a point is equal to average value of four surrounding points. So we can set up a matrix system to solve problems like finding to electric potential.

Example

Two infinitely-long grounded metal plates, at y=0 and y=a, are connected at x=±b by metal strips maintained at constant potential V_0. Find the potential inside the pipe.



i) Usual method:

Differential equation:

$$V(x,y) = (Ae^{kx} + Be^{-kx})(Csin(ky) + Dcos(ky))$$

Boundary conditions:

$$\begin{cases} (i) & V = 0 \; when \; y = 0 \\ (ii) & V = 0 \; when \; y = a \\ (iii) & V = V_0 \; when \; x = b \\ (iv) & V = V_0 \; when \; x = -b \end{cases}$$

Because the symmetry of the system, V(-x,y)=V(x,y), A=B, and absorbing 2A=(A+B) into C and D we have:

$$V(x,y) = (Ae^{kx} + Ae^{-kx})(Csin(ky) + Dcos(ky))$$

$$V(x,y) = 2Acosh(kx)(Csin(ky) + Dcos(ky))$$

$$V(x,y) = cosh(kx)(Csin(ky) + Dcos(ky))$$

Boundary condition (i) and (ii) require D=0 and k=nπ/a, so

$$V(x,y) = Ccosh(n\pi x/a)sin(n\pi y/a)$$

The solution should be linear combination of above equation, so

$$V(x,y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a)\sin(n\pi y/a)$$

Using condition (iii) to find C_n:

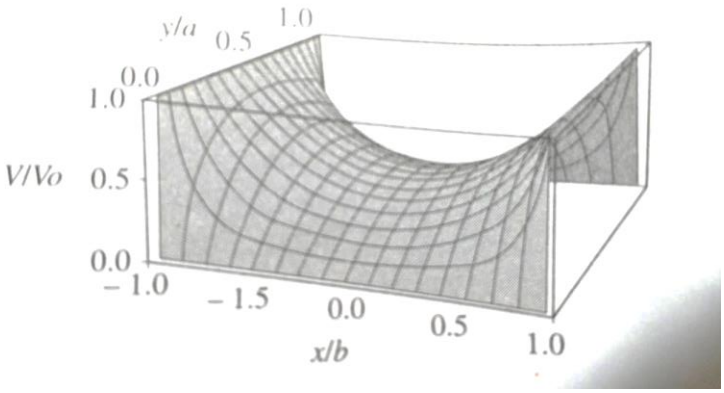
$$V(b,y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right) = V_0$$

By observing the above equation, we can find out that:

$$C_n \cosh\left(\frac{n\pi b}{a}\right) = \begin{cases} 0, if\ n\ is\ even \\ \frac{4V_0}{\pi}, if\ n\ is\ odd \end{cases}$$

So the potential is given by

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n}\frac{\cosh(n\pi x/a)}{\cosh(n\pi x/a)}\sin(n\pi y/a)}$$



Using Maple to find potential of few points:

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> V(x, y) := 4·V0/Pi · sum(1/(2 n − 1) · cosh((2 n − 1)·Pi·x/a)/cosh((2 n − 1)·Pi·b/a)
      ·sin((2 n − 1)·Pi·y/a), n = 1 .. infinity)
> b := 1/2
> a := 1
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> $V0 := 1$

> $evalf\left(V\left(-\frac{1}{4}, \frac{3}{4}\right)\right)$

0.5000000000

> $evalf\left(V\left(0, \frac{3}{4}\right)\right)$

0.3640566637

> $evalf\left(V\left(\frac{1}{4}, \frac{3}{4}\right)\right)$

0.5000000000

> $evalf\left(V\left(-\frac{1}{4}, \frac{1}{2}\right)\right)$

0.6359433360

> $evalf\left(V\left(-0, \frac{1}{2}\right)\right)$

0.5000000000

> $evalf\left(\mathrm{V}\left(\frac{1}{4}, \frac{1}{2}\right)\right)$

0.6359433360

> $evalf\left(V\left(-\frac{1}{4}, \frac{1}{4}\right)\right)$

0.5000000000

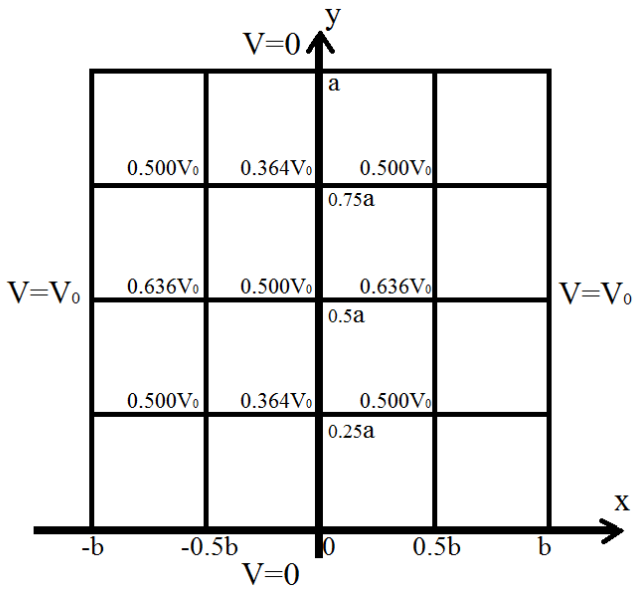
> $evalf\left(V\left(0, \frac{1}{4}\right)\right)$

0.3640566637

> $evalf\left(V\left(\frac{1}{4}, \frac{1}{4}\right)\right)$

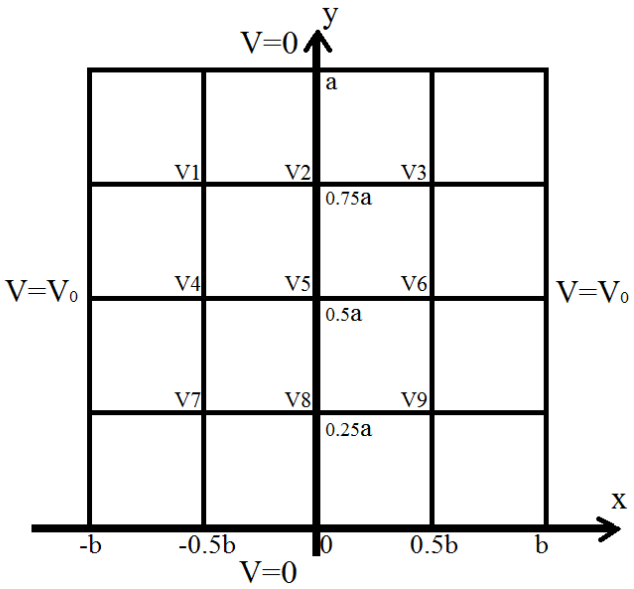
0.5000000000

So we have



ii) Numerical Method

With the same 9 points, set the potential at those points to be V1, V2…V9.



Because the relation

$$4\,u(x_i, y_j) - u(x_{i+1}, y_j) - u(x_{i-1}, y_j) - \left(u(x_i, y_{j+1}) - u(x_i, y_{j-1})\right) = 0$$

we have following equations:

$$\begin{cases} 4V1 = 0 + V2 + V4 + V0 \\ 4V2 = 0 + V3 + V5 + V1 \\ 4V3 = 0 + V0 + V6 + V2 \\ 4V4 = V1 + V5 + V7 + V0 \\ 4V5 = V2 + V6 + V8 + V4 \\ 4V6 = V3 + V0 + V9 + V5 \\ 4V7 = V4 + V8 + 0 + V0 \\ 4V8 = V5 + V9 + 0 + V7 \\ 4V9 = V6 + V0 + 0 + V8 \end{cases}$$

Set up matrix system:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} V1 \\ V2 \\ V3 \\ V4 \\ V5 \\ V6 \\ V7 \\ V8 \\ V9 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

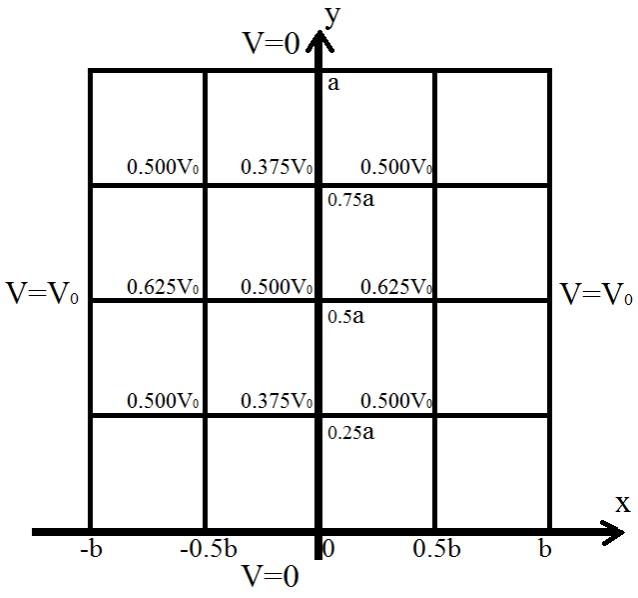
Solved by Maple:

$$> M := \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & 1 \end{bmatrix} :$$

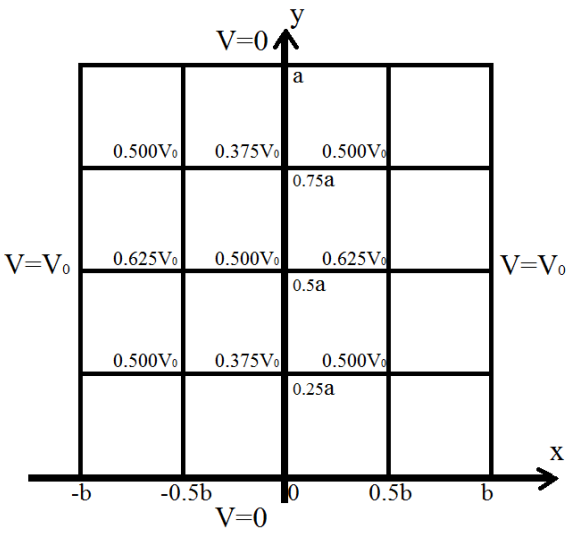
> *evalf*(*LinearAlgebra*:-*ReducedRowEchelonForm*(M))

$$\begin{bmatrix} 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.5000000000 \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.3750000000 \\ 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0.5000000000 \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0.6250000000 \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0.5000000000 \\ 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0.6250000000 \\ 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0.5000000000 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0.3750000000 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0.5000000000 \end{bmatrix}$$

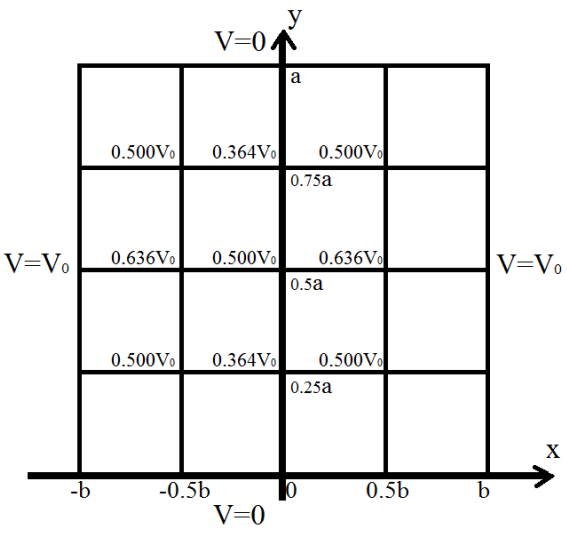
So we have



Analysis and Conclusion



(Potential field by solving DE)



(Potential field using numerical method)

Potentials calculated using numerical method is very close to actual value. If more points are used, the result should be even more accurate. So, when we only need approximate value of few point on an electric potential field (or similar problem) given boundary condition, numerical method can be an efficient way to do that.

Reference

Graves, E. *Numerical Laplace Equation.* Retrieved from Moodle: https://moodle.rose-hulman.edu/pluginfile.php/811430/mod_resource/content/0/NUMERICAL%20LAPLACE%20EQUATION.pdf

Griffiths, D. J. *Introduction to Electrodynamics.*