Quantum Game Theory

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Abstract— Game theory has wide implementation on many fields of study. With more recent development in quantum computation and quantum information, we are able to study games with rule of quantum mechanics. This paper consists two part: in first part, basic knowledge of game theory is briefly recalled, and prisoner's dilemma is introduced; in second part, how quantum version game is differ from classical is discussed, and an analysis quantum version of prisoner's dilemma is presented.

Keywords— game theory; prisoner's dilemma; Nash equilibrium; entanglement;

1. Introduction

Game theory is a well-studied subject that has been widely applied to social science and many other field. On the other hand, quantum computation and quantum communication, which is a more recently developed subject, make it possible to study games with rule of quantum mechanics. A new interdisciplinary subject, quantum game theory, is then established.

Classical game theory is a study on strategic interaction between rational decision-makers. Modern game theory focuses on existence of mixed-strategy equilibriums in two-person zero-sum games, which is proof by John von Neumann[5]. In 1950, the well-known mathematical discussion of the prisoner's dilemma appeared, and around the same time, John Nash developed a criterion for mutual consistency of players' strategies[6][9].

With properties of quantum mechanics, for example, superposition and entanglement, quantum game theory would be far more complex compared with the classical version.

2. Classical game theory

Game theory can be defined as the study of mathematical models of strategic interaction between rational decision-makers[2]. Individuals involved in a game are interrelated, as one's decision is affected by others', and also one's decision affect others'. Therefore, no one in the game is able to control the result completely, and no one is completely isolated from others. In addition, individuals in the game are rational decision-makers, which means they should make decisions that will maximize their expected payoff. Because of interdependency of players, a rational decision must be made based on anticipating other players' action.

In mathematics, most games usually have following three elements:

1. Players: which can be represented as:

$$i = 1, 2, \ldots$$

2. Strategies: each player has multiple strategies, and these strategies form a Strategic Space. For player *i*, if he has *k* pure strategies (a pure strategy determines the action of a player would make for any situation that the player could face, while a mixed strategy gives a probability distribution of pure strategy), then his or her Strategic Space can be represented as:

$$S_i = \{s_{i1}, s_{i2}, \dots, s_{ik}\}$$

Sometimes, Strategic Space can be continuous instead of discrete as well.

3. Payoff: each player's payoff is a function:

$$\$(s), s = \{s_1, s_2, \dots\}$$

where *s* is combination of all player's strategies called Strategy Profile, and s_i is i^{th} player's strategy.

* Some studies may also include elements such as: information, outcome, equilibrium, and etc.

2.1 Game types

There are many ways to classify games into different categories. For example, a game can have its player making choices simultaneously or sequentially. When decisions are made simultaneously, the game is static. On the other hand, when decisions are made in sequence, a subsequent player is able to make a decision based on former players' action, thus the game is dynamic. Also, players in a game can have complete information or incomplete information. Combine the above two ideas, we can get four different types of game, and these four games have four corresponding equilibriums:

	Static	Dynamic
Complete information	Static game of complete information; Nash equilibrium	Dynamic game of complete information; subgame perfect Nash equilibrium
Incomplete information	Static game of incomplete information; Bayesian Nash equilibrium	Dynamic game of incomplete information; perfect Bayesian Nash equilibrium

Table 1. Games and their equilibrium.

In this paper, we will analysis prisoner's dilemma, which is a static game with complete information, so the emphasis will be on Nash equilibrium.

2.2 Nash equilibrium

The Nash equilibrium, is a Strategy Profile of a game involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy.

Formally, in the case of static game with *n* players: $i = 1,2, \dots, n$. i^{th} player has Strategic Space S_i with strategies $s_i \in S_i$. If there is a Strategy Profile $s^* = \{s_1^*, s_2^*, \dots, s_n^*\}$ which satisfy:

$$\forall i, s_i \in Si : \$(s_1^*, \dots, s_i^*, \dots, s_n^*) \geq \$(s_1^*, \dots, s_i, \dots, s_n^*)$$

Such Strategy Profile is a Nash equilibrium.

2.3 Classical prisoner's dilemma

The prisoner's dilemma is an example of a game analyzed in game theory. It was originally framed by Merrill Flood and Melvin Dresher in 1950. Then Albert W. Tucker formalized the game. A slightly modified version game is as follows:

Two criminals are arrested. Each of them cannot communicate with the other. Each player has the choices either to Defect, which is betray the other by testifying that the other committed the crime, or to Cooperate by staying silent. Depending on their choices, each of them receives a certain payoff:

	B: Cooperate	B: Defect
A: Cooperate	(3,3)	(0,5)
A: Defect	(5,0)	(1,1)

Table 2. Payoff matrix for the Prisoners' Dilemma, with first entry denotes the payoff of player A and second entry denotes the second player's. The value of payoff are chosen as in [5].

If both prisoners choose to defect, then each of them will not be better off by changing his or her own strategy. To be specific, originally, each of them has payoff equal to 1, if one of them decided to cooperate in this case, his or her pay off would drop to 0, and therefore no one under this situation would change strategy. Mutual defection thus is a Nash equilibrium for this game.

This game is called 'dilemma' because both of prisoners choosing to cooperate give the result with maximum total payoff, yet, after 'rational' decision making, both would make a choice resulting less payoff than the mutual cooperate case. What if a quantum version of the game is performed, will the dilemma be solved?

3 Quantum game theory[1]

Quantum game theory differ from the classical version in three ways:

1. Superposed initial states: For a classical two-strategy case, one player's choice can be represented by classical bit 0 or 1. While in quantum version, the bit is replaced by a qubit, which is initially prepared in superposition of 0 and 1. Player's choice is corresponded to operation on the qubit.

2. Quantum entanglement of initial states: The set of qubits provided to each player can be initially entangled, so one player's operation on his or her qubit can affect others' qubit, which means one's choice can altering others' expected payoffs of the game.

3. Superposition of strategies: In the quantum case, since the initial state is represented by a qubit, a player choosing a strategy is in analogy with rotating the qubit to a new state, or applying a unitary matrix to the state vector of the qubit. The new state may not be definite, it can still be in superposition of basis states with changed probability amplitudes.

3.1 Quantum prisoner's dilemma

The setup for a quantum version of this game includes: (i) a source of two qubits, one for each player; (ii) a set of instruments which allow each player to manipulate his or her qubit according to the strategy; (iii) a measurement device that measures the final state of the set of qubits, and determine the payoff of each player. And the setup is perfectly known by both players.

The classical strategies D (defect) and C (cooperate) can be assigned to two basis vectors $|D\rangle$ and $|C\rangle$ in the Hilbert space of a two-level system. Then the basis state of combined two qubit system can be written using tensor product, including $|CC\rangle$, $|CD\rangle$, $|DC\rangle$, and $|DD\rangle$, with first entry and second entry refer to player A's and player B's qubits respectively.

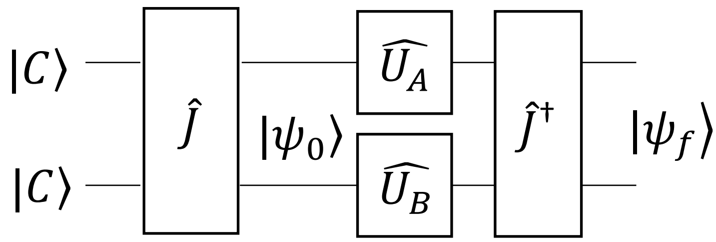

Fig. 1. Setup of the game[5]

The setup is depicted in Fig. 1. The initial state of two qubits is $|\psi_0\rangle = \hat{J}|CC\rangle$, where $\hat{J}$ is a unitary operator that entangle the two qubit. Two players' strategic moves are made through unitary operators $\widehat{U_A}$ and $\widehat{U_B}$. Notice, each player can only operate on his or her own qubit; however, because the entangling of two qubits, one player's action will affect the other. At the end, an operator $\widehat{J^{\dagger}}$, which is the conjugate transpose of the operator $\hat{J}$, is applied to measure the final state.

The reason for $\widehat{J^{\dagger}}$ able to measure the final state is much like what we did in class: we first apply an Hadamard gate to the first qubit, then apply a CNOT gate to make two qubit entangle; at the end, we take the measure by applying a CNOT gate and an Hadamard gate again.

*In fact, if $\hat{J}$ make two qubits entangle completely, the circuit can look like:

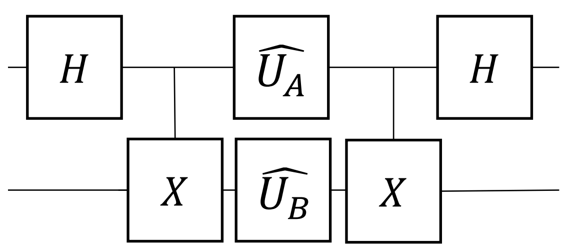

Fig. 2. An analogue of the setup when two qubit are entangled completely

The final state $|\psi_f\rangle$ is given by:

$$|\psi_f\rangle = \widehat{J^{\dagger}}(\widehat{U_A} \otimes \widehat{U_B})\hat{J}|CC\rangle \tag{1}$$

The probability $|\psi_f\rangle$ collapse to classical situations $|CC\rangle$, $|CD\rangle$, $|DC\rangle$, and $|DD\rangle$ can be calculated via inner product:

$$P_{\mu\nu} = \left|\langle \mu\nu|\psi_f\rangle\right|^2 \tag{2}$$

Using payoff matrix in Table 2, players' expected payoff should equal:

$$\begin{aligned} \$_A &= 3P_{cc} + 1P_{DD} + 5P_{DC} + 0P_{CD} \\ \$_B &= 3P_{cc} + 1P_{DD} + 0P_{DC} + 5P_{CD} \end{aligned} \tag{3}$$

According to [5], the above expected payoff restrict the strategic space to a 2-parameter set of unitary matrices:

$$\hat{U}(\theta,\phi) = \begin{pmatrix} e^{i\phi}\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix}, 0 \le \theta \le \pi \; and \; 0 \le \theta \le \pi \tag{4}$$

Cooperate and Defect are just two cases of the operator $\hat{U}$:

$$\begin{aligned} \hat{C} &= \hat{U}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \hat{D} &= \hat{U}(\pi,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{5}$$

The exact form for operator $\hat{J}$ is more complicated. In order to make the game fair, $\hat{J}$ must be symmetric when the two player is interchanged. This condition require:

$$[\hat{J}, \hat{D} \otimes \hat{D}\,] = 0, [\hat{J}, \hat{D} \otimes \hat{C}\,] = 0, [\hat{J}, \hat{C} \otimes \hat{D}\,] = 0 \tag{6}$$

where the square brackets stand for commute of two operator, that is, $[u, v] = u.v - v.u$.

From [5],$\hat{J}$ that satisfy the condition in Eq. (6) has form:

$$\hat{J} = e^{\frac{i\gamma \hat{D}\otimes\hat{D}}{2}} \tag{7}$$

where γ is a real parameter between 0 and $\pi/2$ that tells how much two qubits are entangled. And the matrix exponential can be understand as a power series: $e^X = \sum_{k=1}^{\infty} \frac{1}{k!} X^k$.

Now we have sufficient idea about the circuit in Fig. 1. And we are able to calculate the expected payoff from the perspective of each player: first Plugging Eq. (7) and Eq. (4) into Eq. (1) gives the explicit form of final state of the two-qubit system; then using Eq. (2), probably of the final state collapse to each basis state can be found; finally, substitute these probabilities into Eq. (3). Let us consider two extreme cases of this game when two qubits are completely not entangled and completely entangled:

- when $\gamma = 0$:

For simplicity, reparametrize $\hat{U}$ in a way so that it would only depend on a single parameter: $\widehat{U_A} = \hat{U}(t\pi, 0)\; for\; t \in [0,1]\; and\; \widehat{U_A} = \hat{U}(0, -t\pi/2)\; for\; t \in [-1,0]$ for player A, and similarly, for play B, $\widehat{U_A} = \hat{U}(s\pi, 0)\; for\; s \in [0,1]\; and\; \widehat{U_A} = \hat{U}(0, -s\pi/2)\; for\; s \in [-1,0]$.

The expected payoff for player A is shown below (for detail see Appendix):

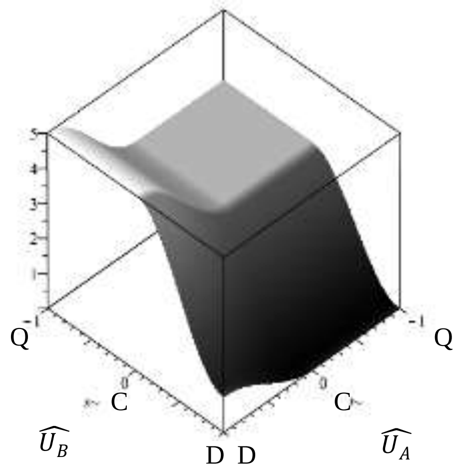

Fig. 3. Player A's expected payoff ($\gamma = 0$).

As shown in Fig. 3, from player A's point of view, for any $\widehat{U_B}$ player B would pick, $\widehat{D}$ always gives maximized payoff. The expected payoff for payer B is symmetric, so a rational decision for player B is also $\widehat{D}$. In addition, by changing one's own stratagem will not lead to increase in payoff. Thus in $\gamma = 0$ case, there is a single Nash equilibrium that is $\widehat{D} \otimes \widehat{D}$.

In fact, when $\gamma = 0$, two qubits in the game are completely not entangled, that is, back to the classical case with mutual defect as Nash equilibrium.

- when $\gamma = \pi/2$:

The expected payoff for player A is shown below:

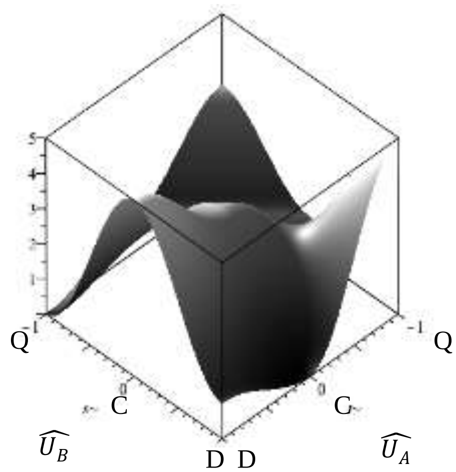

Fig. 3. Player A's expected payoff ($\gamma = \pi/2$).

From player A's respect, if B chooses $\widehat{D}$, the strategy maximize payoff would be a strategy that does not exist in classical game:

$$\hat{Q} = \hat{U}\left(0, \frac{\pi}{2}\right) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (8)$$

But if B chooses $\hat{C}$, the strategy maximize payoff would be $\hat{D}$.

In this case strategy $\hat{D}$ no longer guarantee maximum payoff for any strategy that the other player pick. Indeed $\hat{D} \otimes \hat{D}$ is even not a Nash equilibrium anymore. However, there exist a new Nash equilibrium, that is $\hat{Q} \otimes \hat{Q}$, as $\$_A(\hat{Q}, \hat{Q}) = 3$ and $\$_A(\widehat{U_A}, \hat{Q}) = \cos^2\left(\frac{\theta}{2}\right)(3\sin^2(\phi) + \cos^2(\phi) \leq$ 3 (from the symmetry of the game, same hold for play B).

In this case, when two qubits are completely entangled, the two rational decision maker is able to reach maximum total payoff. The ‘dilemma’ is resolved!

There are also many interesting study regarding two qubits are partially entangled ($0 < \gamma < \pi/2$). As did in [3] and [4], there exist two threshold value for $\gamma_{th1} = \arcsin(\sqrt{1/5}\,)$, $\gamma_{th2} = \arcsin(\sqrt{2/5}\,)$. With low entanglement between two qubit ($0 \leq \gamma \leq \gamma_{th1}$), the Nash equilibrium, $\hat{D} \otimes \hat{D}$, is the same as classical game; with moderate entanglement ($\gamma_{th1} \leq \gamma \leq \gamma_{th2}$), there are two Nash equilibriums $\hat{D} \otimes \hat{Q}$ *and* $\hat{Q} \otimes \hat{D}$; with high entanglement ($\gamma_{th2} \leq \gamma \leq \pi/2$), the situation is close to complete entangled case, and $\hat{Q} \otimes \hat{Q}$ is the Nash equilibrium.

4 Conclusion

In classical prisoner’s dilemma, rational decision-makers fall into the equilibrium that does not maximize the total payoff, while in quantum version, with high enough entanglement between the plays’ qubits, the equilibrium also leads to maximum total payoff. It seems like quantum strategies have superior performance compared with classical strategies, just as what we learned in class that entanglement makes quantum computation more efficient and quantum communication cryptographically safer.

Once more mature quantum circuit is developed, we might expect its implements on decision-making process that resolve problem similar as prisoner’s dilemma and bring about maximum total benefit, such as a quantum voting which might help increase public interest.

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Appendix

Maple code for plot expected payoff

Load packages

> $restart$

> $with(plots):$

> $with(LinearAlgebra):$

Create two real variable

> $assume(s, 'real', t, 'real')$

Pick a value of gamma (strength of entanglement)

> $g := \frac{\text{Pi}}{2}:$

 # g∈(0,Pi/2)

Define basis states

> $c := \begin{bmatrix} 1 \\ 0 \end{bmatrix} : d := \begin{bmatrix} 0 \\ 1 \end{bmatrix}:$

> $cc := KroneckerProduct(c, c) : cd := KroneckerProduct(c, d) : dc := KroneckerProduct(d, c) : dd := KroneckerProduct(d, d):$

Define operator J and its conjugate transpose

> $KroneckerProduct\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

> $J := evalf\left(\sum_{k=0}^{100}\left(\frac{1}{k!}\cdot\left(I\cdot g\cdot\frac{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}{2}\right)^k\right)\right):$

 # 100 terms are used to approximate. For g=0 case, replace J by identity matrix since e^0=I.

> $JJ := HermitianTranspose(J):$

Define UA and UB for case t,s>0 and t,s<0

> $UA1 := \begin{bmatrix} \cos\left(\frac{t\cdot\text{Pi}}{2}\right) & \sin\left(\frac{t\cdot\text{Pi}}{2}\right) \\ -\sin\left(\frac{t\cdot\text{Pi}}{2}\right) & \cos\left(\frac{t\cdot\text{Pi}}{2}\right) \end{bmatrix}:$

> $UA2 := \begin{bmatrix} \exp\left(-\frac{I \cdot t \cdot \text{Pi}}{2}\right) & 0 \\ 0 & \exp\left(\frac{I \cdot t \cdot \text{Pi}}{2}\right) \end{bmatrix}$:

> $UB1 := \begin{bmatrix} \cos\left(\frac{s \cdot \text{Pi}}{2}\right) & \sin\left(\frac{s \cdot \text{Pi}}{2}\right) \\ -\sin\left(\frac{s \cdot \text{Pi}}{2}\right) & \cos\left(\frac{s \cdot \text{Pi}}{2}\right) \end{bmatrix}$:

> $UB2 := \begin{bmatrix} \exp\left(-\frac{I \cdot s \cdot \text{Pi}}{2}\right) & 0 \\ 0 & \exp\left(\frac{I \cdot s \cdot \text{Pi}}{2}\right) \end{bmatrix}$:

Define UA⊗UB

> *UAUB*[1] := *KroneckerProduct*(*UA1*, *UB1*) :
> *UAUB*[2] := *KroneckerProduct*(*UA1*, *UB2*) :
> *UAUB*[3] := *KroneckerProduct*(*UA2*, *UB1*) :
> *UAUB*[4] := *KroneckerProduct*(*UA2*, *UB2*) :

Define the final state psi and expected payoff

>
for *i* **from** 1 **to** 4 **do**
psi[*i*] := *evalf*(*JJ*.*UAUB*[*i*].*J*.*cc*) :
$pcc[i] := |Transpose(cc).\text{psi}[i]|^2$:
$pdd[i] := |Transpose(dd).\text{psi}[i]|^2$:
$pdc[i] := |Transpose(dc).\text{psi}[i]|^2$:
$S[i] := 3\,pcc[i] + pdd[i] + 5\,pdc[i]$:
end do:

Plot payoff function

> *plot1* := *plot3d*(evalf(S[1](1)), *t* = 0 ..1, *s* = 0 ..1) :
> *plot2* := *plot3d*(evalf(S[2](1)), *t* = 0 ..1, *s* =−1 ..0) :
> *plot3* := *plot3d*(evalf(S[3](1)), *t* =−1 ..0, *s* = 0 ..1) :
> *plot4* := *plot3d*(evalf(S[4](1)), *t* =−1 ..0, *s* =−1 ..0) :
> *display*({*plot1*, *plot2*, *plot3*, *plot4*})