Multi-atom cavity quantum electrodynamics

An attempt to realize retardation on the dynamics of entanglement

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The dynamics of two or three atoms interacting with a multi-mode field of a ring cavity is discussed. Concurrence is used as an indicator of the degree of entanglement, through which we are able to study to retardation due to both finite time required for light to travel between atoms and due to atomic interaction. Although the expected retardation is not observed due to inadequate knowledge about parameters, we found adding atoms into the cavity might potentially enhance the entanglement, and sustain the excited state of an already excited atom.

I. Introduction

Entanglement is an interesting phenomenon emerging from non-classical correlations between quantum systems. It is being wildly studied due to its application on quantum computation and quantum communication. There are various methods to generate entanglement between different systems. Atoms can be entangled by slowly passing through a cavity, and the entanglement can be detected by measuring the atomic state[5]. Photons can be entangled in a downconversion process, and polarizations of a photon pair are correlated[6]. As discussed through the semester, for example in the Jaynes-Cummings model, atoms and photons can be entangled as well.

One motivation of studying the retardation on the dynamics of entanglement is its potential application on transferring entanglement between distant quantum systems[7]. Numbers of experiments have successfully created quantum gates which are essential components of quantum networks[8]. Such setup would involve coupling a system with a large environment over distance. Normally, under the Markovian approximation, the entanglement would decay exponentially. However, for certain types of entanglement, the decay is non-exponential, which is known as the sudden death of entanglement (SDE)[9]. In addition to that, with proper conditions, the already destroyed entanglement could be restored, which is known as the sudden birth of entanglement (SBE)[10].

Experimentally, cavities are commonly used to enhance the light-matter interaction as it creates a large coupling strength. In such a case, the system is reversible and non-Markovian, and SBE may be achieved. A conventional way to study the dynamics of such a system uses the master equation approach. For a cavity with multiple atoms, frequently there are a few assumptions: (i) those atoms are separated by large distance such that there is no direct interaction between them; (ii) one atom's influences on another is effectively instantaneously. Such assumptions are safe for a single-mode cavity. When there are more modes in the cavity, the retardation effects become noticeable as the time delay for light propagation between atoms is equivalent to a phase difference. As a result, the distance between atoms is an important parameter in the population and entanglement dynamics. To be specific, whether SDE or SBE can occur or not can depend on the distance.

In this paper, we study the effect of retardation on the population and entanglement dynamics of 2 and 3 two-level atoms in a ring cavity. Using the Schrodinger equation in the interaction picture, we are able to find the time evolution of the population of excitation. Then by using the concurrence as a measure of entanglement, we can also trace the time evolution of entanglement. In the calculation part, we shall show the retardation effects do affect the entanglement between atoms. However, abrupt kinks in the time evolution of the populations and the concurrence corresponding to the time required for the photon to travel between the atoms or loop around the cavity, which is observed in the original paper's plots[1], is not observed in our plots due to insufficient information about parameters. Similarly, the SDE and SBE are not observed in our plots.

Sec. II introduces the model and gives a brief summary for the major result for the two-atom case. Detailed calculation for the three-atom case is in Sec. III, and Sec. IV shows our results.

II. The model

The setup is consisted by identical atoms inside a ring cavity. Distance between atom$_i$ and atom$_j$ is denoted as $x_i - x_j = x_{ij}$ (FIG. 1). The atoms are modeled as two-level systems with ground state $|g_i\rangle$ and excited state $|e_i\rangle$, and those two states are separated by energy $\hbar\omega_a$. The

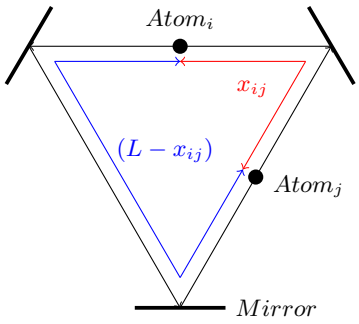

FIG. 1. Schematic diagram of the system. Inside a one-dimensional ring cavity of round trip path L, identical two-level atoms are seperated by distant x_{ij}.

cavity mode frequency are denoted as ω_μ.

A. Summary for two-atom results

The two-atom case are descirbed in detail in [1] and [2]. Also, the overall ideal is similar to that of three-atom case. Therefore only the most important results of the two-atom case are listed here. From both [1] and [2], the probability amplitudes of atoms and field has time evolution as follows:

$$\dot{b_j}(t) = \sum_\mu g_{\mu j} b_\mu(t), j \in \{1, 2\}, \tag{II.1}$$

$$\dot{b_\mu}(t) = -i\Delta_\mu b_\mu(t) - \sum_{j=1}^{2} g^*_{\mu j} b_j(t). \tag{II.2}$$

Here b_j is for atomic state, and b_μ is for field state. $\Delta_\mu = \omega_\mu - \omega_a$ is the detuning of the cavity mode frequency from atomic transition frequency. $g_{\mu j}$ is the coupling constant from dipole approximation of a one dimensional case [4]:

$$\begin{aligned} g_{\mu i} =& g_\mu(\vec{x}_i) \\ =& \frac{\mathcal{E}_\mu}{\hbar}(\vec{d_i} \cdot \hat{e_l}) \mathrm{e}^{i\vec{k}_\mu \cdot \vec{x}_i} \\ =& \sqrt{\frac{\omega_\mu}{2\epsilon_0 \hbar L}}(\vec{d_i} \cdot \hat{e_l}) \mathrm{e}^{i\vec{k}_\mu \cdot \vec{x}_i}. \end{aligned} \tag{II.3}$$

Eq. (II.1-2) can be used directly to study the time evolution of probability amplitude (see Maple code in Appendix for detail). Alternatively, using the set of coupled first order differential equations Eq. (II.1-2), we can have uncoupled second order differential equation

$$\begin{aligned} \dot{b}_j t =& \sum_\mu g_{\mu j} b + \mu(0) \mathrm{e}^{-i\Delta_\mu t} \\ &+ \sum_{j'=1}^{2} \int_0^t dt' \sum_\mu g_{\mu j} g^*_{\mu j'} b_{j'}(t') \mathrm{e}^{-i\Delta_\mu (t-t')}. \end{aligned} \tag{II.4}$$

Using Eq. (II.4) we can perform a numerical integration for the time evolution of probability amplitude as well (see MATLAB code in Appendix for detail).

Concurrence for the two-atom case turned out to be [1][2]

$$C(t) = 2\max\{0, |b_1^*(t) b_2(t)|\}. \tag{II.5}$$

III. Calculation

A. Hamiltonian of the System

For a quantum system comprising of atoms and interacting electromagnetic (EM) fields, the Hamiltonian commonly writes

$$\hat{H} = \hat{H}_a + \hat{H}_f + \hat{H}_{int}. \tag{III.1}$$

In two-level atomic model with the ground state energy offset to zero, the free Hamiltonian of three identical atoms can be expressed as

$$\hat{H}_a = \sum_{j=1}^{3} \hbar \omega_a \hat{\sigma}_{+j} \hat{\sigma}_{-j}, \tag{III.2}$$

where ω_a is the atomic transition frequency and $j \in \{1, 2, 3\}$ labels the atoms. In Fock state representation, the free Hamiltonian of the field (with the constant part of the harmonic oscillator formalism for the EM field dropped) is

$$\hat{H}_f = \hbar \sum_\mu \omega_n \hat{a}^\dagger_\mu \hat{a}_\mu, \tag{III.3}$$

where ω_n is the frequency of a unique cavity mode associated with wave number k_n, with the relation given by $k_n = \omega_n / c$; the script μ denotes the set of wave vector $\mathbf{k}_n$ and polarization $\mathbf{e}_s$ of the field that couples with the atom. In dipole approximation, which models atoms in the presence of an external electric field as an electric dipole and regards the interaction energy as the potential energy stored as the dipole energy, the interaction Hamiltonian is given by

$$\hat{H}_{int} = -\sum_{j=1}^{3} \hat{\mathbf{d}}_j \cdot \hat{\mathbf{E}}(\mathbf{x}_j), \tag{III.4}$$

where $\mathbf{x}_j$ denotes the atom j's position. This follows the classical dipole energy expression except that both the dipole and the electric field terms (i.e., $\hat{\mathbf{d}}$ and $\hat{\mathbf{E}}$) are quantized. Define the matrix elements for atom (dipole) j $\mathbf{d}_j = \langle g_j | \hat{\mathbf{d}} | e_j \rangle$ and $\mathbf{d}^*_j = \langle e_j | \hat{\mathbf{d}} | g_j \rangle$, with the diagonal terms dropped due to their odd parity function nature whose integral over all space must vanish, we have

$$\hat{\mathbf{d}}_j = \mathbf{d}_j \hat{\sigma}_{+j} + \mathbf{d}^*_j \hat{\sigma}_{-j}. \tag{III.5}$$

In the case of a one-dimensional multi-mode cavity with length L, the quantization of the electric field is given by

$$\hat{\mathbf{E}}(\mathbf{x}, t) = i \sum_\mu \varepsilon_\mu \left[\hat{a}_\mu e^{i(\mathbf{k}_n \cdot \mathbf{x} - \omega_n t)} - \hat{a}^\dagger_\mu e^{-i(\mathbf{k}_n \cdot \mathbf{x} - \omega_n t)} \right] \hat{\mathbf{e}}_s, \tag{III.6}$$

where $\varepsilon_\mu \equiv \sqrt{\frac{\hbar \omega_n}{2\epsilon_o L}}$. Further specify the quantized field's parameter as $(\mathbf{x}_j, t)$ since only the fields at where atoms are can make a contribution and the interaction Hamiltonian becomes

$$\begin{aligned} \hat{H}_{int} =& -\sum_{j=1}^{3} (\mathbf{d}_j \hat{\sigma}_{+j} + \mathbf{d}^*_j \hat{\sigma}_{-j}) \\ & i \sum_\mu \varepsilon_\mu \left[\hat{a}_\mu e^{i(\mathbf{k}_n \cdot \mathbf{x}_j - \omega_n t)} - \hat{a}^\dagger_\mu e^{-i(\mathbf{k}_n \cdot \mathbf{x}_j - \omega_n t)} \right] \hat{\mathbf{e}}_s \\ =& -i\hbar \sum_{j=1}^{3} \sum_\mu \frac{\varepsilon_\mu}{\hbar} \left[(\mathbf{d}_j \cdot \hat{\mathbf{e}}_s) \hat{\sigma}_{+j} + (\mathbf{d}^*_j \cdot \hat{\mathbf{e}}_s) \hat{\sigma}_{-j} \right] \\ & \left[\hat{a}_\mu e^{i(\mathbf{k}_n \cdot \mathbf{x}_j - \omega_n t)} - \hat{a}^\dagger_\mu e^{-i(\mathbf{k}_n \cdot \mathbf{x}_j - \omega_n t)} \right]. \end{aligned} \tag{III.7}$$

To proceed, we pull out the time dependence of the operators ($\hat{\sigma}$'s and $\hat{a}$'s). Applying Heisenberg Equation of Motion for $\hat{\sigma}_{+j}$, $\hat{\sigma}_{-j}$, $\hat{a}_\mu$, and $\hat{a}^\dagger_\mu$ separately, we can get a

set of solutions:

$$\hat{a}_\mu(t) = \hat{a}_\mu(0)e^{i\omega_n t}, \tag{III.8}$$

$$\hat{a}^\dagger_\mu(t) = \hat{a}^\dagger_\mu(0)e^{-i\omega_n t}, \tag{III.9}$$

$$\hat{\sigma}_{-j}(t) = \hat{\sigma}_{-j}(0)e^{i\omega_a t}, \tag{III.10}$$

$$\text{and } \hat{\sigma}_{+j}(t) = \hat{\sigma}_{+j}(0)e^{-i\omega_a t}. \tag{III.11}$$

Plug the solution set above into Eq.(III.7) and apply Rotating Wave Approximation (i.e., drop the fast oscillating terms with $e^{i(\omega_a+\omega_\mu)t}$ and $e^{-i(\omega_a+\omega_\mu)t}$), the interaction Hamiltonian takes the form

$$\begin{aligned}\hat{H}_{int} =& -i\hbar\sum_{j=1}^{3}\sum_{\mu}\left[g_\mu(\mathbf{x}_j,t)\hat{\sigma}_{+j}(t)\hat{a}_\mu(t)\right.\\ &\left.- g^*_\mu(\mathbf{x}_j,t)\hat{\sigma}_{-j}(t)\hat{a}^\dagger_\mu(t)\right]\\ =& -i\hbar\sum_{j=1}^{3}\sum_{\mu}\left[g_{\mu j}\hat{\sigma}_{+j}\hat{a}_\mu - g^*_{\mu j}\hat{\sigma}_{-j}\hat{a}^\dagger_\mu\right](t)\end{aligned} \tag{III.12}$$

where $g_{\mu j} = g_\mu(\mathbf{x}_j,t) \equiv \frac{\varepsilon_\mu}{\hbar}(\hat{\mathbf{d}}_j\cdot\hat{\mathbf{e}}_s)e^{i(\mathbf{k}_n\cdot\mathbf{x}_j-\omega_n t)}$ is the Rabi frequency signifying the coupling strength between atom j and cavity mode μ.

To find the time evolution of the system, we switch and seek solutions to the Schrodinger equation in the interaction picture:

$$i\hbar\frac{\partial\left|\Psi(t)\right\rangle}{\partial t} = \hat{H}^I_{int}\left|\Psi(t)\right\rangle, \tag{III.13}$$

where the system described by the state vector $|\Psi(t)\rangle$ (represented in interaction picture) and the superscript of I over $\hat{H}_{int}$ indicates the interaction picture form for Eq. (III.12), which is given by

$$\hat{H}^I_{int} = e^{i\hat{H}_o t/\hbar}\hat{H}_{int}e^{-i\hat{H}_o t/\hbar}, \tag{III.14}$$

where $\hat{H}_o = \hat{H}_a + \hat{H}_f$ given by Eq.(III.2) and Eq.(III.3). Let $|\psi(t)\rangle$ represent the state vector of the system in Schrödinger picture, the transformation is

$$\left|\Psi(t)\right\rangle = e^{i\hat{H}_o t/\hbar}\left|\psi(t)\right\rangle. \tag{III.15}$$

Plug the above-mentioned transformations in Eq.(III.13) and have

$$i\hbar\frac{\partial}{\partial t}\left[e^{i\hat{H}_o t/\hbar}\left|\psi(t)\right\rangle\right] = e^{i\hat{H}_o t/\hbar}\hat{H}_{int}\left|\psi(t)\right\rangle, \tag{III.16}$$

or, equivalently,

$$\begin{aligned}&\frac{\partial}{\partial t}\left[e^{i\hat{H}_o t/\hbar}\left|\psi(t)\right\rangle\right]\\ =& -e^{i\hat{H}_o t/\hbar}\sum_{j=1}^{3}\sum_{\mu}\left[g_{\mu j}\hat{\sigma}_{+j}\hat{a}_\mu - g^*_{\mu j}\hat{\sigma}_{-j}\hat{a}^\dagger_\mu\right](t)\left|\psi(t)\right\rangle.\end{aligned} \tag{III.17}$$

In order to observe retardation effect of the system which is caused by, in classical view, a time delay needed for light to travel from one atom to another, we keep the propagation time scale sufficiently small so that the system is in coherent dynamics described by the Hamiltonian above such that there is no considerations over cavity loss, interaction with the environment, spontaneous emission of fields outside the interacting modes, etc. needed. We also keep the system as simple as possible and allow no more than triple excitation (on atoms and/or on cavity modes) in the system, as if the excitation states in the cavity all come from the photon emission by atom(s).

B. Case I: Single Excitation

Let the Schrödinger-picture state vector for single excitation (with a superscript of I) of the system be

$$\begin{aligned}\left|\psi^{\mathrm{I}}(t)\right\rangle =& b_1(t)\left|e_1g_2g_3\{0\}_\mu\right\rangle + b_2(t)\left|g_1e_2g_3\{0\}_\mu\right\rangle\\ &+ b_3(t)\left|g_1g_2e_3\{0\}_\mu\right\rangle + \sum_\mu b_\mu(t)\left|g_1g_2g_3\{1\}_\mu\right\rangle\\ =&\sum_{j=1}^{3} b_j(t)\left|e_jg_{j+1}g_{j+2}\{0\}_\mu\right\rangle\\ &+\sum_\mu b_\mu(t)\left|g_1g_2g_3\{1\}_\mu\right\rangle,\end{aligned} \tag{III.18}$$

where the atomic index $j \in \{1,2,3\} \cong$ (isomorphic to) $\mathbb{Z}_3$ such that $3+1=1$ define a cycle of 1, 2, and 3; e and g stand for the excited and ground state of the atom, respectively; $\{0\}_\mu$ indicates that all possible cavity modes are in Fock state $|0\rangle$ while $\{1\}_\mu$ indicates that mode μ is in $|1\rangle$ with the rest being in $|0\rangle$. Each cat state is expressed in the product state of the individual states it comprises, e.g., $|e_1g_2g_3\{0\}_\mu\rangle = |e_1\rangle\otimes|g_2\rangle\otimes|g_3\rangle\otimes|\{0\}_\mu\rangle$.

Before proceeding, we transform the state vector in Interaction Picture in accordance with Eq.(III.15) and have

$$\begin{aligned}\left|\Psi(t)^{\mathrm{I}}\right\rangle =& e^{i\hat{H}_o t/\hbar}\left|\psi^{\mathrm{I}}(t)\right\rangle\\ =&\sum_{j=1}^{3} b_j(t)e^{i\omega_a t}\left|e_jg_{j+1}g_{j+2}\{0\}_\mu\right\rangle\\ &+\sum_\mu b_\mu(t)e^{i\omega_\mu t}\left|g_1g_2g_3\{1\}_\mu\right\rangle.\end{aligned} \tag{III.19}$$

Now, offset all the rotation phases by $e^{-i\omega_a t}$ so that we make measurements in the reference frame that rotates in parallel with the atomic transition frequency and have

$$\begin{aligned}\left|\Psi^{\mathrm{I}}(t)\right\rangle =&\sum_{j=1}^{3} b_j(t)\left|e_jg_{j+1}g_{j+2}\{0\}_\mu\right\rangle\\ &+\sum_\mu b_\mu(t)e^{-i\Delta_\mu t}\left|g_1g_2g_3\{1\}_\mu\right\rangle,\end{aligned} \tag{III.20}$$

where $\Delta_\mu \equiv \omega_a - \omega_\mu$.

1. Single-Excitation Solution

Apply Eq. (III.17) onto Eq. (III.18) and we get the left hand side (LHS):

$$\begin{aligned} LHS = &\sum_{j=1}^{3} \dot{b}_j(t) \left|e_j g_{j+1} g_{j+2}\{0\}_\mu\right\rangle \\ &+ \sum_\mu \dot{b}_\mu(t) e^{-i\Delta_\mu t} \left|g_1 g_2 g_3\{1\}_\mu\right\rangle \\ &- \sum_\mu i\Delta_\mu b_\mu(t) e^{-i\Delta_\mu t} \left|g_1 g_2 g_3\{1\}_\mu\right\rangle, \end{aligned} \tag{III.21}$$

and right hand side (RHS):

$$\begin{aligned} RHS = -\Big[&\sum_{j=1}^{3}\sum_\mu g_{\mu j} b_\mu(t) \left|e_j g_{j+1} g_{j+2}\{0\}_\mu\right\rangle \\ &- \sum_{j=1}^{3}\sum_\mu g^*_{\mu j} b_j(t) e^{-i\Delta_\mu t} \left|g_1 g_2 g_3\{1\}_\mu\right\rangle \Big]. \end{aligned} \tag{III.22}$$

Equating LHS to RHS, we have

$$\dot{b}_j(t) = -\sum_\mu g_{\mu j} b_\mu(t) \tag{III.23}$$

$$\dot{b}_\mu(t) = i\Delta_\mu b_\mu(t) + \sum_{j=1}^{3} g^*_{\mu j} b_j(t). \tag{III.24}$$

This is the analytical solution of the $\dot{b}(t)$'s. The ultimate solution to the $b(t)$'s requires further numerical methods.

2. Single-Excitation Concurrence

Define a new basis in terms of the states for two atoms: $|1\rangle = |e_j e_{j+1}\rangle$, $|2\rangle = |e_j g_{j+1}\rangle$, $|3\rangle = |g_j e_{j+1}\rangle$, and $|4\rangle = |g_j g_{j+1}\rangle$. The concurrence between two atoms in this basis is given by [3]:

$$\begin{aligned} C(t) = 2\max\{0, &|\rho_{23}(t)| - \sqrt{\rho_{44}(t)\rho_{11}(t)}, \\ &|\rho_{14}(t)| - \sqrt{\rho_{22}(t)\rho_{33}(t)}\}. \end{aligned} \tag{III.25}$$

To find the concurrence between atoms 1 and 2, we first find the density matrix:

$$\begin{aligned} \rho^{\mathrm{I}} = &\left|\Psi^{\mathrm{I}}(t)\right\rangle \left\langle\Psi^{\mathrm{I}}(t)\right| = \left|\psi^{\mathrm{I}}(t)\right\rangle \left\langle\psi^{\mathrm{I}}(t)\right| \\ = &|b_1(t)|^2 \left|e_1 g_2 g_3\{0\}_\mu\right\rangle \left\langle e_1 g_2 g_3\{0\}_\mu\right| \\ &+ |b_2(t)|^2 \left|g_1 e_2 g_3\{0\}_\mu\right\rangle \left\langle g_1 e_2 g_3\{0\}_\mu\right| \\ &+ |b_3(t)|^2 \left|g_1 g_2 e_3\{0\}_\mu\right\rangle \left\langle g_1 g_2 e_3\{0\}_\mu\right| \\ &+ \sum_\mu |b_\mu(t)|^2 \left|g_1 g_2 g_3\{1\}_\mu\right\rangle \left\langle g_1 g_2 g_3\{1\}_\mu\right| \\ &+ b_1(t) b_2^*(t) \left|e_1 g_2 g_3\{0\}_\mu\right\rangle \left\langle g_1 e_2 g_3\{0\}_\mu\right| \\ &+ b_2(t) b_1^*(t) \left|g_1 e_2 g_3\{0\}_\mu\right\rangle \left\langle e_1 g_2 g_3\{0\}_\mu\right| \\ &+ ND, \end{aligned} \tag{III.26}$$

where ND stands for non-diagonal terms that will be traced out when taking partial trace over atom 3 and cavity fields.

Now take the partial trace over atom 3 and cavity fields on Eq. (III.25) and express it in the basis given above, we have

$$\begin{aligned} \rho^{\mathrm{I}}_{(12)} = &|b_1(t)|^2 |2\rangle\langle 2| + |b_2(t)|^2 |3\rangle\langle 3| \\ &+ \Big[|b_3(t)|^2 + \sum_\mu |b_\mu(t)|^2 \Big] |4\rangle\langle 4| \\ &+ b_1(t) b_2^*(t) |2\rangle\langle 3| + b_2(t) b_1^*(t) |3\rangle\langle 2|. \end{aligned} \tag{III.27}$$

Plug this into Eq. (III.25),

$$\begin{aligned} C^{\mathrm{I}}_{(12)}(t) &= 2\max\{0, |b_1(t) b_2^*(t)|, -|b_1(t)|\,|b_2(t)|\} \\ &= 2\max\{0, |b_1(t) b_2^*(t)|\}. \end{aligned} \tag{III.28}$$

Similarly, for concurrence between atoms 2 and 3, and 3 and 1, we have

$$C^{\mathrm{I}}_{(23)}(t) = 2\max\{0, |b_2(t) b_3^*(t)|\}, \tag{III.29}$$

$$\text{and } C^{\mathrm{I}}_{(31)}(t) = 2\max\{0, |b_3(t) b_1^*(t)|\}. \tag{III.30}$$

The average concurrence for single excitation, therefore, is

$$\left\langle C^{\mathrm{I}}(t)\right\rangle = \frac{1}{3}\left[C^{\mathrm{I}}_{(12)}(t) + C^{\mathrm{I}}_{(23)}(t) + C^{\mathrm{I}}_{(31)}(t)\right]. \tag{III.31}$$

C. Case II: Double Excitation

Following the same vector state notation and the rotating frame as in single excitation, for double excitation, the state vector can be expressed as

$$\begin{aligned} \left|\psi^{\mathrm{II}}(t)\right\rangle = &\sum_{j=1}^{3} b_{jj+1}(t) \left|e_j e_{j+1} g_{j+2}\{0\}_\alpha\right\rangle \\ &+ \sum_{j=1}^{3}\sum_\alpha b_{\alpha j}(t) \left|e_j g_{j+1} g_{j+2}\{1\}_\alpha\right\rangle \\ &+ \sum_{\alpha>\beta} b_{\alpha\beta}(t) \left|g_1 g_2 g_3\{1\}_\alpha\{1\}_\beta\right\rangle \\ &+ \sum_\alpha b_{\alpha\alpha}(t) \left|g_1 g_2 g_3\{2\}_\alpha\right\rangle. \end{aligned} \tag{III.32}$$

Note that $\sum_{\alpha>\beta}$ reads the summation over the index pair $\{\alpha, \beta\}$ such that $\alpha > \beta$.

1. Double-Excitation Solution

Apply Eq.(III.17) on (III.32), we get

$$
\begin{aligned}
&LHS = \\
&\sum_{j=1}^{3}\dot{b}_{jj+1}(t)\left|e_j e_{j+1} g_{j+2}\{0\}_\alpha\right\rangle \\
&+\sum_{j=1}^{3}\sum_{\alpha}\dot{b}_{\alpha j}(t)e^{-i\Delta_\alpha t}\left|e_j g_{j+1} g_{j+2}\{1\}_\alpha\right\rangle \\
&+\sum_{\alpha>\beta}\dot{b}_{\alpha\beta}(t)e^{-i(\Delta_\alpha+\Delta_\beta)t}\left|g_1 g_2 g_3\{1\}_\alpha\{1\}_\beta\right\rangle \\
&+\sum_{\alpha}\dot{b}_{\alpha\alpha}(t)e^{-2i\Delta_\alpha t}\left|g_1 g_2 g_3\{2\}_\alpha\right\rangle \\
&-i\Delta_\alpha\sum_{j=1}^{3}\sum_{\alpha}b_{\alpha j}(t)e^{-i\Delta_\alpha t}\left|e_j g_{j+1} g_{j+2}\{1\}_\alpha\right\rangle \\
&-i(\Delta_\alpha+\Delta_\beta)\sum_{\alpha>\beta}b_{\alpha\beta}(t)e^{-i(\Delta_\alpha+\Delta_\beta)t}\left|g_1 g_2 g_3\{1\}_\alpha\{1\}_\beta\right\rangle \\
&-2i\Delta_\alpha\sum_{\alpha}b_{\alpha\alpha}(t)e^{-2i\Delta_\alpha t}\left|g_1 g_2 g_3\{2\}_\alpha\right\rangle, \qquad \text{(III.33)}
\end{aligned}
$$

and

$$
\begin{aligned}
&RHS = \\
&-\Bigg[\sum_{j=1}^{3}\sum_{\alpha}(g_{\alpha j}b_{\alpha j+1}+g_{\alpha j+1}b_{\alpha j})\left|e_j e_{j+1} g_{j+2}\{0\}_\alpha\right\rangle \\
&+\sum_{j=1}^{3}\Big(\sum_{\alpha>\beta}g_{\beta j}b_{\alpha\beta}+\sum_{\alpha<\beta}g_{\beta j}b_{\beta\alpha}\Big)e^{-i\Delta_\alpha t}\left|e_j g_{j+1} g_{j+2}\{1\}_\alpha\right\rangle \\
&+\sum_{j=1}^{3}\sum_{\alpha}\sqrt{2}g_{\alpha j}b_{\alpha\alpha}e^{-i\Delta_\alpha t}\left|e_j g_{j+1} g_{j+2}\{1\}_\alpha\right\rangle \\
&-\sum_{j=1}^{3}\sum_{\alpha}(g^*_{\alpha j+2}b_{j+2,j}+g^*_{\alpha j+1}b_{jj+1})e^{-i\Delta_\alpha t}\left|e_j g_{j+1} g_{j+2}\{1\}_\alpha\right\rangle \\
&-\sum_{j=1}^{3}\sum_{\alpha>\beta}(g^*_{\beta j}b_{\alpha j}+g^*_{\alpha j}b_{\beta j})e^{-i(\Delta_\alpha+\Delta_\beta)t}\left|g_1 g_2 g_3\{1\}_\alpha\{1\}_\beta\right\rangle \\
&-\sum_{j=1}^{3}\sum_{\alpha}\sqrt{2}g^*_{\alpha j}b_{\alpha j}e^{-2i\Delta_\alpha t}\left|g_1 g_2 g_3\{2\}_\alpha\right\rangle\Bigg]. \qquad \text{(III.34)}
\end{aligned}
$$

Equating LHS to RHS, we get the solutions:

$$
\begin{aligned}
\dot{b}_{jj+1} =& -\sum_{\alpha}(g_{\alpha j}b_{\alpha j+1}+g_{\alpha j+1}b_{\alpha j}) \\
\dot{b}_{\alpha j} =& i\Delta_\alpha b_{\alpha j}+g^*_{\alpha j+2}b_{j+2,j}+g^*_{\alpha j+1}b_{jj+1} \\
&-\sqrt{2}g_{\alpha j}b_{\alpha\alpha}-\sum_{\beta(\beta>\alpha)}g_{\beta j}b_{\beta\alpha}-\sum_{\beta(\beta<\alpha)}g_{\beta j}b_{\alpha\beta} \\
\dot{b}_{\alpha\beta} =& i(\Delta_\alpha+\Delta_\beta)b_{\alpha\beta}+\sum_{j=1}^{3}(g^*_{\beta j}b_{\alpha j}+g^*_{\alpha j}b_{\beta j}) \\
\dot{b}_{\alpha\alpha} =& 2i\Delta_\alpha b_{\alpha\alpha}+\sum_{j=1}^{3}\sqrt{2}g^*_{\alpha j}b_{\alpha j}. \qquad \text{(III.35)}
\end{aligned}
$$

2. Double Excitation Concurrence

Expand the j part for Eq.(III.32), we get

$$
\begin{aligned}
\left|\psi^{\mathrm{II}}(t)\right\rangle =& b_{12}(t)\left|e_1 e_2 g_3\{0\}_\alpha\right\rangle \\
&+b_{23}(t)\left|g_1 e_2 e_3\{0\}_\alpha\right\rangle \\
&+b_{31}(t)\left|e_1 g_2 e_3\{0\}_\alpha\right\rangle \\
&+\sum_{\alpha}b_{\alpha 1}(t)\left|e_1 g_2 g_3\{1\}_\alpha\right\rangle \\
&+\sum_{\alpha}b_{\alpha 2}(t)\left|g_1 e_2 g_3\{1\}_\alpha\right\rangle \\
&+\sum_{\alpha}b_{\alpha 3}(t)\left|g_1 g_2 e_3\{1\}_\alpha\right\rangle \\
&+\sum_{\alpha>\beta}b_{\alpha\beta}(t)\left|g_1 g_2 g_3\{1\}_\alpha\{1\}_\beta\right\rangle \\
&+\sum_{\alpha}b_{\alpha\alpha}(t)\left|g_1 g_2 g_3\{2\}_\alpha\right\rangle. \qquad \text{(III.36)}
\end{aligned}
$$

Have the density matrix of it and trace over atom 3 and the cavity modes, we have

$$
\begin{aligned}
\rho^{\mathrm{II}}_{(12)} =& |b_{12}(t)|^2\left|1\right\rangle\left\langle 1\right| + (|b_{31}(t)|^2+\sum_{\alpha}|b_{\alpha 1}(t)|^2)\left|2\right\rangle\left\langle 2\right| \\
&+(|b_{23}(t)|^2+\sum_{\alpha}|b_{\alpha 2}(t)|^2)\left|3\right\rangle\left\langle 3\right| \\
&+\Big(\sum_{\alpha}|b_{\alpha 3}(t)|^2+\sum_{\alpha>\beta}|b_{\alpha\beta}(t)|^2 \\
&+\sum_{\alpha}|b_{\alpha\alpha}(t)|^2\Big)\left|4\right\rangle\left\langle 4\right| \\
&+(b_{31}(t)b^*_{23}(t)+\sum_{\alpha}b_{\alpha 1}(t)b^*_{\alpha 2}(t))\left|2\right\rangle\left\langle 3\right| \\
&+(b_{23}(t)b^*_{31}(t)+\sum_{\alpha}b_{\alpha 2}(t)b^*_{\alpha 1}(t))\left|3\right\rangle\left\langle 2\right|. \qquad \text{(III.37)}
\end{aligned}
$$

Plug this into Eq. (III.25), we have

$$
\begin{aligned}
&C^{\mathrm{II}}_{(12)}(t) \\
=&2\max\Bigg\{0, \Big|b_{31}(t)b^*_{23}(t)+\sum_{\alpha}b_{\alpha 1}(t)b^*_{\alpha 2}(t)\Big| \\
&-\left|b_{(12)}(t)\right|\sqrt{\sum_{\alpha}|b_{\alpha 3}(t)|^2+\sum_{\alpha>\beta}|b_{\alpha\beta}(t)|^2+\sum_{\alpha}|b_{\alpha\alpha}(t)|^2}\Bigg\}. \qquad \text{(III.38)}
\end{aligned}
$$

Similarly, for concurrence between atom 2 and 3,

$$
\begin{aligned}
&C^{\mathrm{II}}_{(23)}(t) \\
=&2\max\Bigg\{0, \Big|b_{12}(t)b^*_{31}(t)+\sum_{\alpha}b_{\alpha 2}(t)b^*_{\alpha 3}(t)\Big| \\
&-\left|b_{(23)}(t)\right|\sqrt{\sum_{\alpha}|b_{\alpha 1}(t)|^2+\sum_{\alpha>\beta}|b_{\alpha\beta}(t)|^2+\sum_{\alpha}|b_{\alpha\alpha}(t)|^2}\Bigg\}, \qquad \text{(III.39)}
\end{aligned}
$$

and for atom 3 and 1,

$$\begin{aligned}&C^{\mathrm{II}}_{(31)}(t)\\ =&2\max\Bigg\{0, \left|b_{23}(t)b^*_{12}(t)+\sum_\alpha b_{\alpha 3}(t)b^*_{\alpha 1}(t)\right|\\ &-\left|b_{(31)}(t)\right|\sqrt{\sum_\alpha|b_{\alpha 2}(t)|^2+\sum_{\alpha>\beta}|b_{\alpha\beta}(t)|^2+\sum_\alpha|b_{\alpha\alpha}(t)|^2}\Bigg\}.\end{aligned} \tag{III.40}$$

Lastly, the average concurrence for double excitation writes:

$$\left\langle C^{\mathrm{II}}(t)\right\rangle=\frac{1}{3}\left[C^{\mathrm{II}}_{(12)}(t)+C^{\mathrm{II}}_{(23)}(t)+C^{\mathrm{II}}_{(31)}(t)\right]. \tag{III.41}$$

D. Case III: Triple Excitation

Following the previous frame and notation conventions, we have the triple excitation state vector:

$$\begin{aligned}\left|\psi^{\mathrm{III}}(t)\right\rangle =&b_{123}(t)\left|e_1e_2e_3\{0\}_\alpha\right\rangle+\sum_{j=1}^{3}\sum_\alpha b_{\alpha jj+1}(t)\left|e_je_{j+1}g_{j+2}\{1\}_\alpha\right\rangle\\ &+\sum_{j=1}^{3}\sum_\alpha b_{\alpha\alpha j}(t)\left|e_jg_{j+1}g_{j+2}\{2\}_\alpha\right\rangle+\sum_{j=1}^{3}\sum_{\alpha>\beta} b_{\alpha\beta j}(t)\left|e_jg_{j+1}g_{j+2}\{1\}_\alpha\{1\}_\beta\right\rangle\\ &+\sum_{\alpha>\beta>\gamma} b_{\alpha\beta\gamma}(t)\left|g_1g_2g_3\{1\}_\alpha\{1\}_\beta\{1\}_\gamma\right\rangle+\sum_{\alpha>\beta} b_{\alpha\beta\beta}(t)\left|g_1g_2g_3\{1\}_\alpha\{2\}_\beta\right\rangle\\ &+\sum_{\alpha>\beta} b_{\alpha\alpha\beta}(t)\left|g_1g_2g_3\{2\}_\alpha\{1\}_\beta\right\rangle+\sum_\alpha b_{\alpha\alpha\alpha}(t)\left|g_1g_2g_3\{3\}_\alpha\right\rangle.\end{aligned} \tag{III.42}$$

1. Triple-Excitation Solution

Apply Eq.(III.17) onto Eq.(III.42), we get

$$\begin{aligned}LHS =&\dot b_{123}(t)\left|e_1e_2e_3\{0\}_\alpha\right\rangle+\sum_{j=1}^{3}\sum_\alpha \dot b_{\alpha jj+1}(t)e^{-i\Delta_\alpha t}\left|e_je_{j+1}g_{j+2}\{1\}_\alpha\right\rangle\\ &+\sum_{j=1}^{3}\sum_\alpha \dot b_{\alpha\alpha j}(t)e^{-2i\Delta_\alpha t}\left|e_jg_{j+1}g_{j+2}\{2\}_\alpha\right\rangle+\sum_{j=1}^{3}\sum_{\alpha>\beta}\dot b_{\alpha\beta j}(t)e^{-i(\Delta_\alpha+\Delta_\beta)t}\left|e_jg_{j+1}g_{j+2}\{1\}_\alpha\{1\}_\beta\right\rangle\\ &+\sum_{\alpha>\beta>\gamma}\dot b_{\alpha\beta\gamma}(t)e^{-i(\Delta_\alpha+\Delta_\beta+\Delta_\gamma)t}\left|g_1g_2g_3\{1\}_\alpha\{1\}_\beta\{1\}_\gamma\right\rangle+\sum_{\alpha>\beta}\dot b_{\alpha\beta\beta}(t)e^{-i(\Delta_\alpha+2\Delta_\beta)t}\left|g_1g_2g_3\{1\}_\alpha\{2\}_\beta\right\rangle\\ &+\sum_{\alpha>\beta}\dot b_{\alpha\alpha\beta}(t)e^{-i(2\Delta_\alpha+\Delta_\beta)t}\left|g_1g_2g_3\{2\}_\alpha\{1\}_\beta\right\rangle+\sum_\alpha\dot b_{\alpha\alpha\alpha}(t)e^{-3i\Delta_\alpha t}\left|g_1g_2g_3\{3\}_\alpha\right\rangle\\ &-i\Delta_\alpha\sum_{j=1}^{3}\sum_\alpha b_{\alpha jj+1}(t)e^{-i\Delta_\alpha t}\left|e_je_{j+1}g_{j+2}\{1\}_\alpha\right\rangle-2i\Delta_\alpha\sum_{j=1}^{3}\sum_\alpha b_{\alpha\alpha j}(t)e^{-2i\Delta_\alpha t}\left|e_jg_{j+1}g_{j+2}\{2\}_\alpha\right\rangle\\ &-i(\Delta_\alpha+\Delta_\beta)\sum_{j=1}^{3}\sum_{\alpha>\beta} b_{\alpha\beta j}(t)e^{-i(\Delta_\alpha+\Delta_\beta)t}\left|e_jg_{j+1}g_{j+2}\{1\}_\alpha\{1\}_\beta\right\rangle\\ &-i(\Delta_\alpha+\Delta_\beta+\Delta_\gamma)\sum_{\alpha>\beta>\gamma} b_{\alpha\beta\gamma}(t)e^{-i(\Delta_\alpha+\Delta_\beta+\Delta_\gamma)t}\left|g_1g_2g_3\{1\}_\alpha\{1\}_\beta\{1\}_\gamma\right\rangle\\ &-i(\Delta_\alpha+2\Delta_\beta)\sum_{\alpha>\beta} b_{\alpha\beta\beta}(t)e^{-i(\Delta_\alpha+2\Delta_\beta)t}\left|g_1g_2g_3\{1\}_\alpha\{2\}_\beta\right\rangle\\ &-i(2\Delta_\alpha+\Delta_\beta)\sum_{\alpha>\beta} b_{\alpha\alpha\beta}(t)e^{-i(2\Delta_\alpha+\Delta_\beta)t}\left|g_1g_2g_3\{2\}_\alpha\{1\}_\beta\right\rangle\\ &-3i\Delta_\alpha\sum_\alpha b_{\alpha\alpha\alpha}(t)e^{-3i\Delta_\alpha t}\left|g_1g_2g_3\{3\}_\alpha\right\rangle.\end{aligned} \tag{III.43}$$

The RHS is

$$
\begin{aligned}
RHS = -\Bigg[& \sum_{j=1}^{3}\sum_{\alpha} g_{\alpha j} b_{\alpha j+1,j+2}(t)\,|e_1e_2e_3\{0\}_\alpha\rangle \\
& + \sum_{j=1}^{3}\sum_{\alpha} \sqrt{2}(g_{\alpha j} b_{\alpha\alpha j+1}(t) + g_{\alpha j+1} b_{\alpha\alpha j}(t)) e^{-i\Delta_\alpha t}\,|e_j e_{j+1} g_{j+2}\{1\}_\alpha\rangle \\
& + \sum_{j=1}^{3}\sum_{\alpha>\beta} (g_{\beta j} b_{\alpha\beta j+1}(t) + g_{\beta j+1} b_{\alpha\beta j}(t)) e^{-i\Delta_\alpha t}\,|e_j e_{j+1} g_{j+2}\{1\}_\alpha\rangle \\
& + \sum_{j=1}^{3}\sum_{\alpha<\beta} (g_{\beta j} b_{\beta\alpha j+1}(t) + g_{\beta j+1} b_{\beta\alpha j}(t)) e^{-i\Delta_\alpha t}\,|e_j e_{j+1} g_{j+2}\{1\}_\alpha\rangle \\
& + \sum_{j=1}^{3}\sum_{\alpha>\beta>\gamma} g_{\gamma j} b_{\alpha\beta\gamma}(t) e^{-i(\Delta_\alpha+\Delta_\beta)t}\,|e_j g_{j+1} g_{j+2}\{1\}_\alpha\{1\}_\beta\rangle \\
& + \sum_{j=1}^{3}\sum_{\alpha>\gamma>\beta} g_{\gamma j} b_{\alpha\gamma\beta}(t) e^{-i(\Delta_\alpha+\Delta_\beta)t}\,|e_j g_{j+1} g_{j+2}\{1\}_\alpha\{1\}_\beta\rangle \\
& + \sum_{j=1}^{3}\sum_{\gamma>\alpha>\beta} g_{\gamma j} b_{\gamma\alpha\beta}(t) e^{-i(\Delta_\alpha+\Delta_\beta)t}\,|e_j g_{j+1} g_{j+2}\{1\}_\alpha\{1\}_\beta\rangle \\
& + \sum_{j=1}^{3}\sum_{\alpha>\beta} \sqrt{2}(g_{\beta j} b_{\alpha\beta\beta}(t) + g_{\alpha j} b_{\alpha\alpha\beta}(t)) e^{-i(\Delta_\alpha+\Delta_\beta)t}\,|e_j g_{j+1} g_{j+2}\{1\}_\alpha\{1\}_\beta\rangle \\
& + \sum_{j=1}^{3}\Big(\sum_{\alpha<\beta} g_{\beta j} b_{\beta\alpha\alpha}(t) + \sum_{\alpha>\beta} g_{\beta j} b_{\alpha\alpha\beta}(t)\Big) e^{-2i\Delta_\alpha t}\,|e_j g_{j+1} g_{j+2}\{2\}_\alpha\rangle \\
& + \sum_{j=1}^{3}\sum_{\alpha} \sqrt{3} g_{\alpha j} b_{\alpha\alpha\alpha}(t) e^{-2i\Delta_\alpha t}\,|e_j g_{j+1} g_{j+2}\{2\}_\alpha\rangle \\
& - \sum_{j=1}^{3}\sum_{\alpha} g^*_{\alpha j+2} b_{123}(t) e^{-i\Delta_\alpha t}\,|e_j e_{j+1} g_{j+2}\{1\}_\alpha\rangle \\
& - \sum_{j=1}^{3}\sum_{\alpha} \sqrt{2}(g^*_{\alpha j+1} b_{\alpha jj+1}(t) + g^*_{\alpha j+2} b_{\alpha j+2,j}(t)) e^{-2i\Delta_\alpha t}\,|e_j g_{j+1} g_{j+2}\{2\}_\alpha\rangle \\
& - \sum_{j=1}^{3}\sum_{\alpha>\beta} (g^*_{\beta j+1} b_{\alpha jj+1}(t) + g^*_{\beta j+2} b_{\alpha j+2,j}(t)) e^{-i(\Delta_\alpha+\Delta_\beta)t}\,|e_j g_{j+1} g_{j+2}\{1\}_\alpha\{1\}_\beta\rangle \\
& - \sum_{j=1}^{3}\sum_{\alpha>\beta} (g^*_{\alpha j+1} b_{\beta jj+1}(t) + g^*_{\alpha j+2} b_{\beta j+2,j}(t)) e^{-i(\Delta_\alpha+\Delta_\beta)t}\,|e_j g_{j+1} g_{j+2}\{1\}_\alpha\{1\}_\beta\rangle \\
& - \sum_{j=1}^{3}\sum_{\alpha} \sqrt{3} g^*_{\alpha j} b_{\alpha\alpha j}(t) e^{-3i\Delta_\alpha t}\,|g_1 g_2 g_3\{3\}_\alpha\rangle \\
& - \sum_{j=1}^{3}\sum_{\alpha>\beta} g^*_{\beta j} b_{\alpha\alpha j}(t) e^{-i(2\Delta_\alpha+\Delta_\beta)t}\,|g_1 g_2 g_3\{2\}_\alpha\{1\}_\beta\rangle \\
& - \sum_{j=1}^{3}\sum_{\alpha>\beta} g^*_{\alpha j} b_{\beta\beta j}(t) e^{-i(\Delta_\alpha+2\Delta_\beta)t}\,|g_1 g_2 g_3\{1\}_\alpha\{2\}_\beta\rangle \\
& - \sum_{j=1}^{3}\sum_{\alpha>\beta>\gamma} (g^*_{\gamma j} b_{\alpha\beta j}(t) + g^*_{\beta j} b_{\alpha\gamma j}(t) + g^*_{\alpha j} b_{\beta\gamma j}(t)) e^{-i(\Delta_\alpha+\Delta_\beta+\Delta_\gamma)t}\,|g_1 g_2 g_3\{1\}_\alpha\{2\}_\beta\{1\}_\gamma\rangle \\
& - \sum_{j=1}^{3}\sum_{\alpha>\beta} \sqrt{2} g^*_{\alpha j} b_{\alpha\beta j}(t) e^{-i(2\Delta_\alpha+\Delta_\beta)t}\,|g_1 g_2 g_3\{2\}_\alpha\{1\}_\beta\rangle \\
& - \sum_{j=1}^{3}\sum_{\alpha>\beta} \sqrt{2} g^*_{\beta j} b_{\alpha\beta j}(t) e^{-i(\Delta_\alpha+2\Delta_\beta)t}\,|g_1 g_2 g_3\{1\}_\alpha\{2\}_\beta\rangle \Bigg].
\end{aligned}
\tag{III.44}
$$

Equating LHS to RHS, we get:

$$
\begin{aligned}
\dot{b}_{123}(t) =& -\sum_{j=1}^{3}\sum_{\alpha} g_{\alpha j} b_{\alpha j+1,j+2}(t) \\
\dot{b}_{\alpha jj+1}(t) =& i\Delta_\alpha b_{\alpha jj+1}(t) - \sqrt{2}(g_{\alpha j} b_{\alpha\alpha j+1}(t) + g_{\alpha j+1} b_{\alpha\alpha j}(t)) \\
&- \sum_{\beta(\alpha>\beta)} (g_{\beta j} b_{\alpha\beta j+1}(t) + g_{\beta j+1} b_{\alpha\beta j}(t)) - \sum_{\beta(\alpha<\beta)} (g_{\beta j} b_{\beta\alpha j+1}(t) + g_{\beta j+1} b_{\beta\alpha j}(t)) \\
&+ g^*_{\alpha j+2} b_{123}(t) \\
\dot{b}_{\alpha\alpha j}(t) =& 2i\Delta_\alpha b_{\alpha\alpha j}(t) - \big(\sum_{\beta(\alpha<\beta)} g_{\beta j} b_{\beta\alpha\alpha}(t) + \sum_{\beta(\alpha>\beta)} g_{\beta j} b_{\alpha\alpha\beta}(t)\big) \\
&- \sqrt{3} g_{\alpha j} b_{\alpha\alpha\alpha}(t) + \sqrt{2}(g^*_{\alpha j+1} b_{\alpha jj+1}(t) + g^*_{\alpha j+2} b_{\alpha j+2,j}(t)) \\
\dot{b}_{\alpha\beta j}(t) =& i(\Delta_\alpha + \Delta_\beta) b_{\alpha\beta j}(t) - \sum_{\gamma(\alpha>\beta>\gamma)} g_{\gamma j} b_{\alpha\beta\gamma}(t) - \sum_{\gamma(\alpha>\gamma>\beta)} g_{\gamma j} b_{\alpha\gamma\beta}(t) \\
&- \sum_{\gamma(\gamma>\alpha>\beta)} g_{\gamma j} b_{\gamma\alpha\beta}(t) - \sqrt{2}(g_{\beta j} b_{\alpha\beta\beta}(t) + g_{\alpha j} b_{\alpha\alpha\beta}(t)) \\
&+ g^*_{\beta j+1} b_{\alpha jj+1}(t) g^*_{\beta j+2} b_{\alpha j+2,j}(t) + g^*_{\alpha j+1} b_{\beta jj+1}(t) + g^*_{\alpha j+2} b_{\beta j+2,j}(t) \\
\dot{b}_{\alpha\beta\gamma}(t) =& i(\Delta_\alpha + \Delta_\beta + \Delta_\gamma) b_{\alpha\beta\gamma}(t) + \sum_{j=1}^{3} (g^*_{\gamma j} b_{\alpha\beta j}(t) + g^*_{\beta j} b_{\alpha\gamma j}(t) + g^*_{\alpha j} b_{\beta\gamma j}(t)) \\
\dot{b}_{\alpha\beta\beta}(t) =& i(\Delta_\alpha + 2\Delta_\beta) b_{\alpha\beta\beta}(t) + \sum_{j=1}^{3} g^*_{\alpha j} b_{\beta\beta j}(t) + \sum_{j=1}^{3} \sqrt{2} g^*_{\beta j} b_{\alpha\beta j}(t) \\
\dot{b}_{\alpha\alpha\beta}(t) =& i(2\Delta_\alpha + \Delta_\beta) b_{\alpha\alpha\beta}(t) + \sum_{j=1}^{3} g^*_{\beta j} b_{\alpha\alpha j}(t) + \sum_{j=1}^{3} \sqrt{2} g^*_{\alpha j} b_{\alpha\beta j}(t) \\
\dot{b}_{\alpha\alpha\alpha}(t) =& 3i\Delta_\alpha b_{\alpha\alpha\alpha}(t) + \sum_{j=1}^{3} \sqrt{3} g^*_{\alpha j} b_{\alpha\alpha j}(t).
\end{aligned}
\tag{III.45}
$$

2. *Triple-Excitation Concurrence*

As conventionally, take the partial trace over atom 3 and cavity fields and write down the reduced density matrix from Eq.(III.42) in the basis introduced previously:

$$
\begin{aligned}
\rho^{\mathrm{III}}_{(12)}(t) =& |b_{123}(t)|^2 |1\rangle\langle 1| + \sum_{\alpha} |b_{\alpha 12}(t)|^2 |1\rangle\langle 1| \\
&+ \sum_{\alpha} \left[|b_{\alpha\alpha 3}(t)|^2 + |b_{\alpha\alpha\alpha}(t)|^2\right] |4\rangle\langle 4| + \sum_{\alpha>\beta>\gamma} |b_{\alpha\beta\gamma}(t)|^2 |4\rangle\langle 4| \\
&+ \sum_{\alpha>\beta} \left[|b_{\alpha\beta 3}(t)|^2 + |b_{\alpha\beta\beta}(t)|^2 + |b_{\alpha\alpha\beta}(t)|^2\right] |4\rangle\langle 4| \\
&+ \Big[\sum_{\alpha} b_{\alpha 31}(t) b^*_{\alpha 23}(t) + \sum_{\alpha} b_{\alpha\alpha 1}(t) b^*_{\alpha\alpha 2}(t) + \sum_{\alpha>\beta} b_{\alpha\beta 1}(t) b^*_{\alpha\beta 2}(t)\Big] |2\rangle\langle 3| \\
&+ N_{12},
\end{aligned}
\tag{III.46}
$$

where the N_{12} term stands for the non-contributing parts to the concurrence between atoms 1 and 2. Plug into Eq.(III.25) we have the reduced concurrence (since ρ_{14} should vanish in $\rho^{\mathrm{III}}_{(12)}$) for atoms 1 and 2:

$$
C^{\mathrm{III}}_{(12)}(t) = 2\max\{0, \left|\rho^{\mathrm{III}}_{(12)23}(t)\right| - \sqrt{\rho^{\mathrm{III}}_{(12)44}(t)\rho^{\mathrm{III}}_{(12)11}(t)}\},
\tag{III.47}
$$

where

$$\begin{aligned}
\rho^{\mathrm{III}}_{(12)23}(t) =& \sum_{\alpha} b_{\alpha 31}(t) b^{*}_{\alpha 23}(t) + \sum_{\alpha} b_{\alpha\alpha 1}(t) b^{*}_{\alpha\alpha 2}(t) + \sum_{\alpha>\beta} b_{\alpha\beta 1}(t) b^{*}_{\alpha\beta 2}(t), \\
\rho^{\mathrm{III}}_{(12)11}(t) =& |b_{123}(t)|^2 + \sum_{\alpha} |b_{\alpha 12}(t)|^2, \\
\text{and } \rho^{\mathrm{III}}_{(12)44}(t) =& \sum_{\alpha} \left[|b_{\alpha\alpha 3}(t)|^2 + |b_{\alpha\alpha\alpha}(t)|^2 \right] + \sum_{\alpha>\beta>\gamma} |b_{\alpha\beta\gamma}(t)|^2 \\
& + \sum_{\alpha>\beta} \left[|b_{\alpha\beta 3}(t)|^2 + |b_{\alpha\beta\beta}(t)|^2 + |b_{\alpha\alpha\beta}(t)|^2 \right].
\end{aligned} \tag{III.48}$$

Similarly, for Concurrence between atoms 2 and 3:

$$C^{\mathrm{III}}_{(23)}(t) = 2\max\{0, \left|\rho^{\mathrm{III}}_{(23)23}(t)\right| - \sqrt{\rho^{\mathrm{III}}_{(23)44}(t)\rho^{\mathrm{III}}_{(23)11}(t)}\}, \tag{III.49}$$

where

$$\begin{aligned}
\rho^{\mathrm{III}}_{(23)23}(t) =& \sum_{\alpha} b_{\alpha 12}(t) b^{*}_{\alpha 31}(t) + \sum_{\alpha} b_{\alpha\alpha 2}(t) b^{*}_{\alpha\alpha 3}(t) + \sum_{\alpha>\beta} b_{\alpha\beta 2}(t) b^{*}_{\alpha\beta 3}(t), \\
\rho^{\mathrm{III}}_{(23)11}(t) =& |b_{123}(t)|^2 + \sum_{\alpha} |b_{\alpha 23}(t)|^2, \\
\text{and } \rho^{\mathrm{III}}_{(23)44}(t) =& \sum_{\alpha} \left[|b_{\alpha\alpha 1}(t)|^2 + |b_{\alpha\alpha\alpha}(t)|^2 \right] + \sum_{\alpha>\beta>\gamma} |b_{\alpha\beta\gamma}(t)|^2 \\
& + \sum_{\alpha>\beta} \left[|b_{\alpha\beta 1}(t)|^2 + |b_{\alpha\beta\beta}(t)|^2 + |b_{\alpha\alpha\beta}(t)|^2 \right].
\end{aligned} \tag{III.50}$$

And for atoms 3 and 1,

$$C^{\mathrm{III}}_{(31)}(t) = 2\max\{0, \left|\rho^{\mathrm{III}}_{(31)23}(t)\right| - \sqrt{\rho^{\mathrm{III}}_{(31)44}(t)\rho^{\mathrm{III}}_{(31)11}(t)}\}, \tag{III.51}$$

where

$$\begin{aligned}
\rho^{\mathrm{III}}_{(31)23}(t) =& \sum_{\alpha} b_{\alpha 23}(t) b^{*}_{\alpha 12}(t) + \sum_{\alpha} b_{\alpha\alpha 3}(t) b^{*}_{\alpha\alpha 1}(t) + \sum_{\alpha>\beta} b_{\alpha\beta 3}(t) b^{*}_{\alpha\beta 1}(t), \\
\rho^{\mathrm{III}}_{(31)11}(t) =& |b_{123}(t)|^2 + \sum_{\alpha} |b_{\alpha 31}(t)|^2, \\
\text{and } \rho^{\mathrm{III}}_{(31)44}(t) =& \sum_{\alpha} \left[|b_{\alpha\alpha 2}(t)|^2 + |b_{\alpha\alpha\alpha}(t)|^2 \right] + \sum_{\alpha>\beta>\gamma} |b_{\alpha\beta\gamma}(t)|^2 \\
& + \sum_{\alpha>\beta} \left[|b_{\alpha\beta 2}(t)|^2 + |b_{\alpha\beta\beta}(t)|^2 + |b_{\alpha\alpha\beta}(t)|^2 \right].
\end{aligned} \tag{III.52}$$

Finally, the average Concurrence for triple excitation is

$$\left\langle C^{\mathrm{III}}(t) \right\rangle = \frac{1}{3} \left[C^{\mathrm{III}}_{(12)}(t) + C^{\mathrm{III}}_{(23)}(t) + C^{\mathrm{III}}_{(31)}(t) \right]. \tag{III.53}$$

IV. Results

The general features of concurrence is already discussed on the analytical section of the project. This section discusses the numerical calculation of transient behaviour of populations and concurrence. Due to inadequate information about some of the constants like dipole matrix element ($\vec{d_j}$), vacuum resonant frequency of the central resonant mode (Ω_o) and atomic wavelength (λ_α) in the project paper, we cannot reciprocate the results in the paper in exact form and thus, cannot discuss all the features of the project. With these limitations, we can only discuss the single-excitation case with zero inter-atomic spacing.

A. Effects of retardation on the population dynamics

The transient behaviour of the population dynamics is a key concept towards understanding the transient behaviour of the concurrence. Thus, it is important to understand the effect of retardation on the population dynamics before beginning the effects on the concurrence.

However, we are interested in the numerical solution of the problem and thus, some initial conditions are required. The set of coupled Eq. (III.23) and (III.24) describes the population dynamics for single-excitation case, and we assume that the atoms were initially prepared (i.e. at time t=0) in a product state

$$|\psi(0)\rangle = |e_1\rangle \otimes |g_2\rangle \otimes |\{0\}_\mu\rangle \equiv |e_1 g_2 \{0\}_\mu\rangle \,.$$

Mathematically, this assumption means that the probability amplitude of first atom being in excited state at time t=0 is 1, and rest of the probability amplitudes (including photon being in cavity mode) is 0 i.e. $b_1(0) = 1$ and $b_2(0) = b_3(0) = b_\mu(0) = 0$.

We are interested in comparing the results between two atoms and three atoms case and analyze the major differences. FIG. 2(a) and (b) show the transient probabilities of case of 2-atom with no separation and case of 3-atom with no separation respectively. At t=0, the photon is completely absorbed by the first atom as our initial condition suggests. As time increases, the photon is distributed between atoms and cavity modes till maximum distribution occurs at some time slightly less than 2μs. The distribution is repetitive in certain time interval. We used 99 different frequency modes for photon in our numerical calculation and there should be uneven distribution of photon for such different frequency modes because of the varying value of detuning for each mode. Due to interference effect, the probability distribution should not be periodic as we can see in project paper's result where the transient probability is never 1 except at t=0 and decreases exponentially with time. There might be something wrong with our numerical code and the obtained result looks like a single mode case where the distribution of photon is occurring between the atoms and a mode of the cavity.

The major difference between the two atom and the three atom case is that the first atom never completely deexcites, and the second and the third atom never completely excites in three atoms case wheres both of the atoms excites and deexcites with 100% probability in two atoms case. Since the same maximum distribution (around 28%) is happening in both two and three atom case, the distribution can be quantified to be more in three atom case.

To see the immediate evolution of excitation, we can consider the atomic dynamics in terms of the collective Dicke states:

$$\begin{aligned} |g\rangle &= |g_1\rangle \otimes |g_2\rangle \,, \\ |s\rangle &= \frac{1}{\sqrt{2}}(|e_1\rangle \otimes |g_2\rangle + |g_1\rangle \otimes |e_2\rangle), \\ |\alpha\rangle &= \frac{1}{\sqrt{2}}(|e_1\rangle \otimes |g_2\rangle - |g_1\rangle \otimes |e_2\rangle). \end{aligned} \tag{IV.1}$$

The excitation probability for symmetric $|s\rangle$ and antisymmetric $|\alpha\rangle$ states are:

$$\begin{aligned} |b_s(t)|^2 &= \frac{1}{2}\{|b_1(t)|^2 + |b_2(t)|^2 + 2Re[b_1(t)b_2^*(t)]\}, \\ |b_\alpha(t)|^2 &= \frac{1}{2}\{|b_1(t)|^2 + |b_2(t)|^2 - 2Re[b_1(t)b_2^*(t)]\} \end{aligned} \tag{IV.2}$$

Due to lack of calculation for three atoms, we cannot compare our results of two atoms with three atoms. But we are just interested in general overview of two atoms for this case.

FIG. 2(c) shows the transient excitation probabilities for both symmetric and antisymmetric Dicke states. At t=0, both of the states are equally distributed. But as time increases and reached to a value slightly less than 2μs, the symmetric Dicke state reaches to it's minimum value i.e. zero and antismmetric Dicke state also reached to it's minimum value of around 38%. The reminiscent value of antisymmetric Dicke state (or zero value of symmetric Dicke state) suggests the reminiscent of electromagnetically induced transparency or decoherence free subspace.

B. Effect of retardation on entanglement

As we know concurrence is the measure of maximal entanglement of a system, it's evolution will give us how the maximal entanglement of the system is evolving with time. Eq. (III.38-40) gives the concurrence between two atoms whereas Eq. (III.41) gives the average concurrence for three atoms.

The behaviour of distribution of photon between atoms and cavity as discussed in previous subsection can be quantified by concurrence. At t=0, the photon was completely absorbed by the first atom resulting in zero concurrence. At time slightly less than 2μs, there was maximum distribution of photons for two atoms case. We can see a maximum value of concurrence (50%) at that time. Similarly for three atoms case, the concurrence is 58%.

The slope of the concurrence curve is steeper in two atoms case than three atoms case which means that the distribution of photon in the system occurs rapidly in two atoms case than in three atoms case. But the peak value of concurrence is larger in three atoms case implying the photon gets more distributed in three atoms case compared to two atoms case.

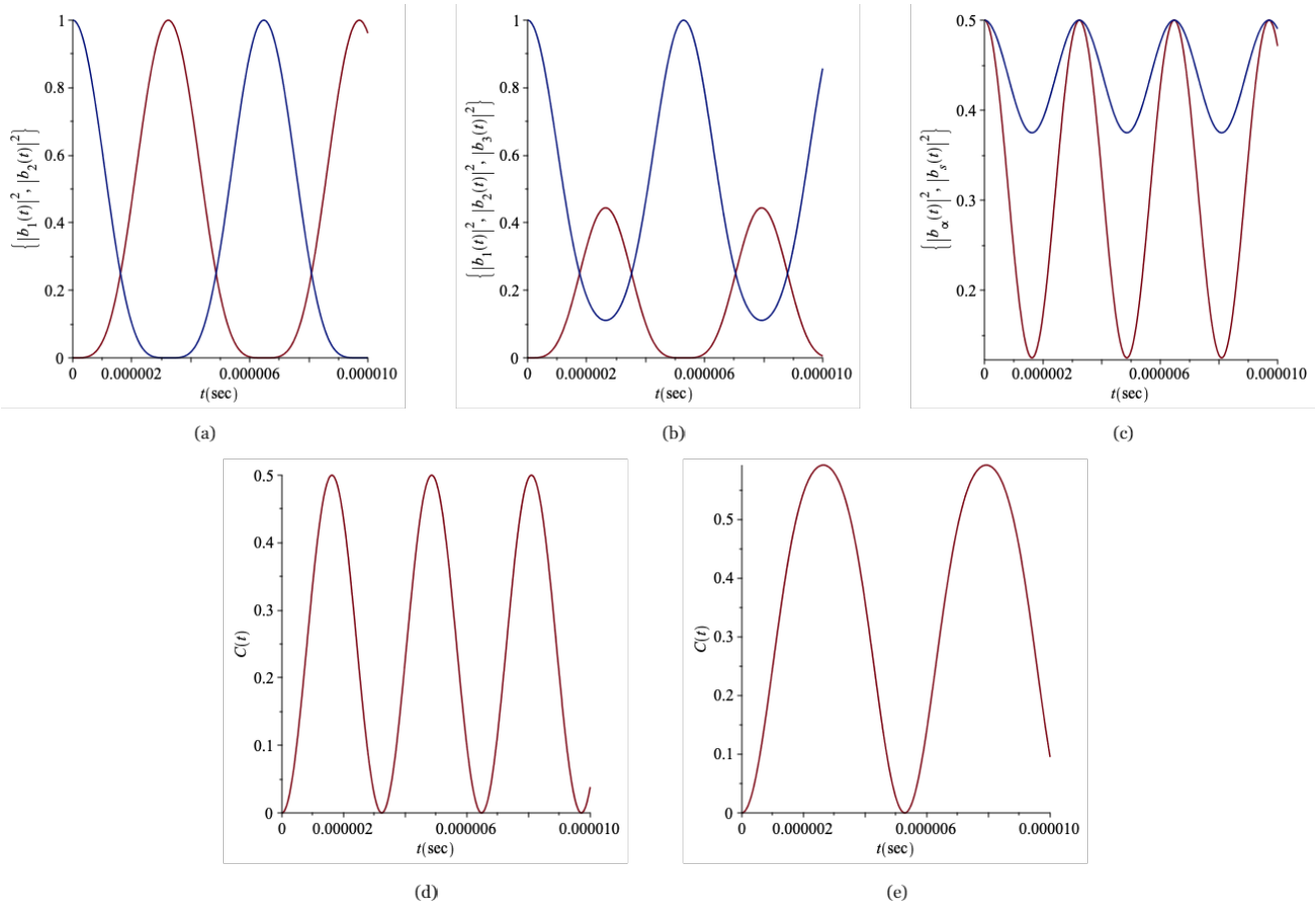

FIG. 2. (a) The transient probability of first (blue) and second (red) atom being excited with the time. (b) The transient probability of first (blue), second (red) and third (red) atom being excited with the time (the second the third atom curve overlapped). (c) The transient excitation probability for symmetric Dicke state (red) with time and the transient excitation probability for antisymmetric Dicke state (blue) with time. (d) The evolution of concurrence between two atoms with time. (e) The evolution of average concurrence between three atoms with time. *Parameters used for those plots: constants are all in SI unit, i.e. $c = 2.998 \times 10^8 m/s, \hbar = 1.054 \times 10^{-34} m^2 \cdot kg/s, \epsilon_0 = 8.854 \times 10^{-12} m^{-3} kg^{-1} s^4 A^2$; transition dipole moment is assumed to be the product of the electron charge and the Bohr radius $d = e \times a_0 = 1.602 \times 10^{-19} C \times 5.2917 \times 10^{-11} m$; detuning Δ_μ is assumed to be integer multiple of 2π, i.e. $\Delta_\mu = 2\pi\mu$.

V. Conclusion

Although we couldn't reciprocate the results of the paper exactly, the extended result we got for 3 atoms is significant (in comparison to the 2 atoms result from the same code). It suggests that the concurrence of the system increases with increment in no. of atoms in the system. In addition to that, when the system goes from 2 atoms to many, the correlation between the atoms happens such that complete deexcitation of first atom and complete excitation of other atoms cannot happen in all times.

Appendix

In the case of 2 atoms with single excitation and without separation between them, MATLAB code for numerical integration for the probability amplitude can be:

```
%using eq15 in [1]
clear all;close all;clc

%constant
%(we are not using correct parameters)
omegaa=6960*pi;
hbar=1.0545718*10^-34;
epsilon=8.854*10^12;
c=2.998*10^8;
x1=0;
x2=999*2.87*10^(-4);
tvec=linspace(0,4,100);
deltat=tvec(2)-tvec(1);
d=(1.602*10^(-19))*(5.2917*10^(-11));

%initialization
b1(1)=1;
b2(1)=0;
for a=0:99
    bu(a+1,1)=0;
end

%evolution
for t=1:99 %time
```

```
    for mu=0:99 %mode
        g1(mu+1)=sqrt(2*pi*mu/(2*epsilon*hbar*c))...
        *exp(1i*2*pi*mu*x1/c)*d;%gmu1
        g2(mu+1)=sqrt(2*pi*mu/(2*epsilon*hbar*c))...
        *exp(1i*2*pi*mu*x2/c)*d;%gmu2
        e(mu+1)=exp(-1i*(2*pi*mu-omegaa)...
        *tvec(t));%phase due to time
    end
    ddb1=-sum(g1.*conj(g1).*e)*b1(t)...
        -sum(g1.*conj(g2).*e)*b2(t);
    ddb2=-sum(g2.*conj(g1).*e)*b1(t)...
        -sum(g2.*conj(g2).*e)*b2(t);
    %"second derivative" - part being int in eq15
    db1=ddb1*deltat*t;
    db2=ddb2*deltat*t;
    %first derivative
    b1(t+1)=b1(t)+db1*deltat;
    b2(t+1)=b2(t)+db2*deltat;
    %evolution of pop after a time step
end

%plot
plot(tvec,abs(b1).^2);
hold on
plot(tvec,abs(b2).^2);
```

In the same case, using Maple for analytical solution:

```
restart
%constants (not correct)
A := 5.2917*(Pi/(2.998*(1.0545718*10^(-34)*8.85418)
*10^(-12)*10^8)*1.602)*10^(-19)*10^(-11);

%differential equation from eq15 in [1]
eq1 := diff(diff(b1(t), t), t) =
-A*(sum(u*b1(t), u = 1 .. 99)
+sum(u*b2(t), u = 1 .. 99));
eq2 := diff(diff(b2(t), t), t) =
-A*(sum(u*b1(t), u = 1 .. 99)
+sum(u*b2(t), u = 1 .. 99));

%initial conditions
ic := (D(b1))(0) = 0, (D(b2))(0) = 0,
b1(0) = 1, b2(0) = 0;

%solution
dsolve({eq1, eq2, ic})
b1 := 1/2+(1/2)*cos(10*sqrt(9418606766)*t)
b2 := -1/2+(1/2)*cos(10*sqrt(9418606766)*t)

%plot
plot({b1^2, b2^2}, t = 0 .. 0.1e-4)
```

---

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